Separable solutions of some quasilinear equations with source reaction

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Abstract We study the existence of singular solutions to the equation $-div(|Du|^{p-2}Du) = |u|^{q-1}u$ under the form $u(r,\theta) = r^{-\beta}\omega(\theta)$, $r > 0, \theta \in S^{N-1}$. We prove the existence of an exponent q below which no positive solutions can exist. If the dimension is 2 we use a dynamical system approach to construct solutions.

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1 Introduction

Let $p > 1$, $q > 0$ and Ω be an open subset of $\mathbb{R}^N$. If $u \in C(\overline{\Omega} \setminus \{0\}) \cap C^1(\Omega)$ is a solution of

$$div\left(|Du|^{p-2} Du\right) + |u|^{q-1} u = 0 \tag{1.1}$$

in Ω which vanishes on $\partial\Omega \setminus \{0\}$, it may develop a boundary singularity at $x = 0$. The description of such boundary singularities should play a key role in the understanding of the boundary behaviour of solutions of these equations (see [3] for the equation with an absorption reaction term and [6] for the old results in the case $p = 2$). The interesting range for exponent q corresponds to a dominent reaction term, that is $q > p - 1$, which will be always assumed. If $\Omega = \mathbb{R}^N_+$ is a half space, or more generally a cone with vertex 0 (and this is most often the asymptotic form of the domain Ω at 0), a natural way to look for specific solutions of (1.1) is to look for solutions which can be written under the form

$$u(x) = u(r,\sigma) = r^{-\beta}\omega(\sigma) \quad r > 0,\, \sigma \in S^{N-1}. \tag{1.2}$$

For equations with an absorption nonlinearity

$$div\left(|Du|^{p-2} Du\right) - |u|^{q-1} u = 0, \tag{1.3}$$

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such separable solutions are the key-stone elements for describing the boundary singularities of solutions of (1.3), as it is shown in [3]. It is expected that such will be the case for (1.1), even if the full theory will be much more difficult in particular due to the absence of comparison principle and simple a priori estimates of Keller-Osserman type. However, for such a solution, it is straightforward to check that

$$\beta = \frac{p}{q+1-p} := \beta_q, \tag{1.4}$$

and ω must be a solution of

$$-\nabla'.\left(\left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{p/2-1}\nabla'\omega\right) - |\omega|^{q-1}\omega = \lambda_{q,p}\left(\beta_q^2\omega^2 + |\nabla'\omega|^2\right)^{p/2-1}\omega, \tag{1.5}$$

in S^{N-1}, where ∇' is the covariant gradient on S^{N-1}, $\nabla'.$ the divergence operator acting on vector fields on S^{N-1} and

$$\lambda_{q,p} = \beta_q(q\beta_q - N).$$

When $p = 2$, $\beta_q = 2/(q-1)$ and (1.5) becomes

$$-\Delta'\omega - |\omega|^{q-1}\omega = \lambda_{q,2}\omega, \tag{1.6}$$

where Δ' is the Laplace-Beltrami operator on S^{N-1} and

$$\lambda_{q,2} = \frac{2}{q-1}\left(\frac{2q}{q-1} - N\right).$$

If S is a subdomain of S^{N-1}, equation (1.6), considered in S, is the Euler-Lagrange variation of the functional

$$I(\psi) = \int_S \left(\frac{1}{2}|\nabla\psi|^2 + \frac{\lambda_{q,2}}{2}\psi^2 - \frac{1}{q+1}|\psi|^{q+1}\right)d\sigma. \tag{1.7}$$

For any $1 < q < (N+1)/(N-3)$ (any $q > 1$ if $N = 2$ or 3) this functional satisfies the Palais-Smale condition. Furthermore, if $\lambda_{q,2} < \lambda_{S,2}$, ($\lambda_{S,2}$ is the first eigenvalue of $-\Delta'$ in $W_0^{1,2}(S)$), Ambrosetti-Rabinowitz theorem [1] or Pohozaev fibration method [12], [13] apply and yield to the existence of non-trivial positive solutions to (1.6) in S vanishing on ∂S; while if $\lambda_{q,2} \geq \lambda_{S,2}$ no such solution exists.

When $p \neq 2$, equation (1.5) cannot be associated to any functional defined on S^{N-1}, except if $q = q_c = (N(p-1)+p)/(N-p)$ (the critical Sobolev exponent -1 for $W^{1,p}$, when $N > p$); therefore, finding functions satisfying it is not straitforward. Besides the constant solutions which exist as soon as $q\beta_q < N$, it is not easy to find other solutions of this equation. However a classical method to construct signed solutions is to start from a fundamental simplicial domain $S \subset S^{N-1}$, corresponding to a tesselation of the sphere associated to a finite group of isometries generated by symmetries through hyperplanes, to look for one positive solution of (1.5) in S vanishing on ∂S, and then to extend it by reflexions accross ∂S. It is natural to associate to S the spherical p-harmonic spectral equation on S which is to find a couple (β, ϕ) where $\beta > 0$, $\phi \in C^1(\bar{S})$ is a positive solution of

$$\begin{cases} -\nabla'.\left(\left(\beta^2\phi^2 + |\nabla'\phi|^2\right)^{p/2-1}\nabla'\phi\right) = \lambda\left(\beta^2\phi^2 + |\nabla'\phi|^2\right)^{p/2-1}\phi \quad \text{in S} \\ \phi = 0 \quad \text{in } \partial\text{S}, \end{cases} \tag{1.8}$$

and $\lambda = \beta\left(\beta(p-1) + p - N\right)$. Equivalently, this means that the function $v(r,\sigma) = r^{-\beta}\phi(\sigma)$ is a positive p-harmonic function into the cone $C_S = \{(r,\sigma) : r > 0, \sigma \in S\}$ which vanishes on ∂S. Given a smooth subdomain $S \subset S^{N-1}$, it is proved in [15], following the method of Tolksdorff [14], that there exists a couple $(\beta, \phi) = (\beta_S, \phi_S)$, where β_S is unique and ϕ_S up to an homothethy, such that (1.8) holds. Denoting

$$\lambda_S = \beta_S\left(\beta_S(p-1) + p - N\right),$$

the couple (ϕ_S, λ_S) is the natural generalization of the first eigenfunction and eigenvalue of the Laplace-Beltrami operator in $W_0^{1,2}(S)$ since $\lambda_S = \lambda_{S,2}$ when $p = 2$. Our first theorem is a non-existence which extends the one already mentioned in the case $p = 2$.

Theorem 1. *Let $S \subset S^{N-1}$ be a smooth subdomain. If $\beta_q \geq \beta_S$ there exists no positive solution of (1.5) in S which vanishes on ∂S.*

Apart the case $p = 2$, the existence counterpart of this theorem is not known in arbitrary dimension, except if $q = q_c$ where (1.5) is the Euler-Lagrange equation of the functional

$$J(\psi) = \int_S \left(\frac{1}{p}\left(\beta_{q_c}^2\psi^2 + |\nabla'\psi|^2\right)^{p/2} - \frac{1}{q_c+1}|\psi|^{q_c+1}\right)d\sigma. \tag{1.9}$$

An application of the already mentioned variational methods leads to an existence result.

If $N = 2$ the problem of finding solutions of (1.1) under the form (1.2) can be completely solved using dynamical systems methods. In order to point out a richer class of phenomena, we shall imbed this problem into the more general class of quasilinear equations with a potential, authorizing the value $p = 1$.

$$div\left(|Du|^{p-2}Du\right) + |u|^{q-1}u - \frac{c}{|x|^p}|u|^{p-2}u = 0 \tag{1.10}$$

in $\mathbb{R}^2 \setminus \{0\}$, with $q > p - 1 \geq 0$ and $c \in \mathbb{R}$. Looking again for solutions under the form (1.2), we see clearly that β must be equal to β_q, while ω is any 2π-periodic solution of

$$\begin{aligned} \frac{d}{d\sigma}\left[\left(\beta_q^2\omega^2 + \left(\frac{d\omega}{d\sigma}\right)^2\right)^{p-2)/2}\frac{d\omega}{d\sigma}\right] + \lambda_q\left[\beta_q^2\omega^2 + \left(\frac{d\omega}{d\sigma}\right)^2\right]^{(p-2)/2}\omega \\ +|\omega|^{q-1}\omega - c|\omega|^{p-2}\omega = 0, \end{aligned} \tag{1.11}$$

where

$$\lambda_q = \beta_q\left(q\beta_q - 2\right) = \beta_q\left(p - 2 + (p-1)\beta_q\right). \tag{1.12}$$

Set

$$c_q = \beta_q^{p-2}\lambda_q = p^{p-1}\frac{(p-2)q+2(p-1)}{(q+1-p)^p}, \tag{1.13}$$

If $c \leq c_q$, the only constant solution is the zero function, while if $c > c_q$, there exist two other constant solutions $\pm(c-c_q)^{1/(q+1-p)}$. Then we can describe the set

$$\mathcal{F} = \pm\mathcal{E}^+ \cup \mathcal{E}$$

of nonzero solutions of (1.11) on S^1, where $\mathcal{E}^+$ is the set of nonnegative solutions, and $\mathcal{E}$ the set of sign changing solutions. Our main result is the following:

Theorem 2. *Assume $p > 1$, $q > p-1$.*
(i)

$$\mathcal{E} = \bigcup_{\substack{k\in\mathbb{N},\\ k=k_q}}^{\infty} \left\{\omega_k(.+\psi) : \psi \in S^1\right\}, \tag{1.14}$$

where ω_k is a function with least period $2\pi/k$, where $k_q = 1$ if $c \geq c_q$; if $c < c_q$, then k_q is the smallest positive integer such that $k_q > M_q$, where

$$M_q = \frac{\pi\beta_q^{1-p}}{2\displaystyle\int_0^{\pi/2} \frac{1+(p-1)\tan^2\theta}{\beta_q^p(p-1)\tan^2\theta + c_q - c\cos^{p-2}\theta}d\theta}. \tag{1.15}$$

(ii) If $c \leq c_q$, then $\mathcal{E}^+ = \varnothing$. If $0 < c - c_q \leq \beta_q^{p-1}/p$, then $\mathcal{E}^+ = \left\{(c-c_q)^{1/(q+1-p)}\right\}$. If $c - c_q > \beta_q^{p-1}/p$, then $\mathcal{E}^+$ contains a set of the form

$$\left\{(c-c_q)^{1/(q+1-p)}\right\} \cup \bigcup_{\substack{k\in\mathbb{N},\\ k=1}}^{k_q^+} \left\{\omega_k^+(.+\psi) : \psi \in S^1\right\}, \tag{1.16}$$

where ω_k^+ is a positive function with least period $2\pi/k$, and k_q^+ is the largest integer smaller than $(p\beta_q^{1-p}(c-c_q))^{1/2}$.

As a consequence we can prove the existence counterpart of Theorem 1 in dimension 2

Corollary 1. *Let $N = 2$ and S be any angular sector of S^1. Then there exists a positive solution of (1.5) vanishing at the two end points of S if and only if $\beta_q < \beta_S$. Furthermore this solution is unique. In particular, existence holds for any sector if $p < 2$ and $q \geq 2(p-1)/(2-p)$.*

The case $p = 1$ appears as a limiting case of the preceding one. In that case we observe that u is a positive solution of (1.10) if and only if $v = u^q$ is a solution of the same equation relative to $q = 1$,

$$div\left(|Dv|^{-1} Dv\right) + v - \frac{c}{|x|} = 0. \tag{1.17}$$

The initial case $c = 0$ is easily treated while the case $c \neq 0$, that we shall analyse in full generality is much richer and delicate.

Theorem 3. *Assume $p = 1$ and $q > 0$.*
(i) If $c \neq 0$, or $c = 0$ and $q > 1$, then $\mathcal{E} = \varnothing$. If $c = 0$ and $q \leq 1$ then

$$\mathcal{E} = \left\{\omega_0(. + \psi) : \psi \in S^1\right\},$$

where $\sigma \mapsto \omega_0(\sigma) := 2^{1/q} \left|\sin\sigma\right|^{1/q-1} \sin\sigma$ is a C^1 solution of (1.11).
(ii) If $c \leq -1$, then $\mathcal{E}^+ = \varnothing$. If $-1 < c < 0$, then $\mathcal{E}^+ = \left\{(c+1)^{1/q}\right\}$. If $c > 0$, then

$$\mathcal{E}^+ = \left\{(c+1)^{1/q}\right\} \cup \bigcup_{\substack{k\in\mathbb{N},\\ k=k_1}}^{k_2} \left\{\omega_k^+(. + \psi) : \psi \in S^1\right\},$$

where ω_k^+ is a positive function with least period $2\pi/k$, and k_2 is the largest integer strictly smaller than $(c+1)^{1/2}$, and k_1 is the smallest integer greater than $\pi/2\int_0^{\pi/2}\sqrt{\frac{\cos\theta}{\cos\theta+2c}}d\theta$. If $c = 0$, then

$$\mathcal{E}^+ = \{1\} \cup \bigcup_{K\in(0,1)} \left\{\omega_K^+(. + \psi) : \psi \in S^1\right\} \cup \begin{cases} \varnothing & \text{if } q \geq 1 \\ \left\{\omega_0^+(. + \psi) : \psi \in S^1\right\} & \text{if } q < 1 \end{cases}$$

where

$$\omega_K^+ = \left(\sqrt{1 - K^2\sin^2\sigma} - K\cos\sigma\right)^{1/q}, \quad \text{and} \quad \omega_0^+ = (2\left|\sin\sigma\right|)^{1/q} \quad \forall\sigma \in S^1.$$

A striking phenomenon is the existence of a 2-parameter family of solutions when $c = 0$.

Our paper is organized as follows: 1- Introduction. 2- The N-dimensional case. 3- The 2-dim dynamical system. 4- The case $p > 1$. 5-The case $p = 1$.

2 The N-dimensional case

2.1 The spherical p-harmonic spectral problem

Let $q > p - 1 > 0$ and S be a smooth connected domain on S^{N-1} and C_S be the cone with vertex 0 generated by S. We look for positive solutions of the p-Laplace equation

$$-\operatorname{div}(|Du|^{p-2}Du) = u^q, \tag{2.1}$$

in $C_S \setminus \{(0)\}$ vanishing on $\partial C_S \setminus \{(0)\}$, under the form

$$u(r, \sigma) = r^{-\beta}\omega(\sigma), \tag{2.2}$$

then $\beta = p/(q+1-p) = \beta_q$ and ω solves

$$\left\{\begin{array}{c} -\nabla'.\left((\beta_q^2\omega^2 + |\nabla'\omega|^2)^{(p-2)/2}\nabla'\omega\right) - \omega^q = \lambda_{q,p}(\beta_q^2\omega^2 + |\nabla'\omega|^2)^{(p-2)/2}\omega \text{ in } S \\ \omega = 0 \text{ on } \partial S. \end{array}\right. \tag{2.3}$$

where

$$\lambda_{q,p} = \beta_q(q\beta_q - N).$$

We denote by β_S the exponent corresponding to the first spherical singular p-harmonic function and by ϕ_S the corresponding function. This means that $\beta_S > 0$ and $u(r,\sigma) = r^{-\beta_S}\phi_S(\sigma)$ is p-harmonic in $C_S \setminus \{(0)\}$ and vanishes on $\partial C_S \setminus \{(0)\}$. Furthermore $\phi = \phi_S > 0$ and satisfies

$$\left\{\begin{array}{c} -\nabla'.\left((\beta_S^2\phi^2 + |\nabla'\phi|^2)^{(p-2)/2}\nabla'\phi\right) = \lambda_S(\beta_S^2\phi^2 + |\nabla'\phi|^2)^{(p-2)/2}\phi \text{ in } S \\ \phi = 0 \text{ on } \partial S. \end{array}\right. \tag{2.4}$$

where

$$\lambda_S = \beta_S(\beta_S(p-1) + p - N).$$

If $\varphi \in C^1(S)$, we denote

$$\mathcal{T}(\varphi) = -\nabla'.\left((\beta_q^2\varphi^2 + |\nabla'\varphi|^2)^{(p-2)/2}\nabla'\varphi\right) - \lambda_{q,p}(\beta_q^2\varphi^2 + |\nabla'\varphi|^2)^{(p-2)/2}\varphi \tag{2.5}$$

We recall that (β_S, ϕ_S) is unique up to an homothety upon ϕ. Furthermore ϕ_S is positive in S, $\partial\phi_S/\partial\nu < 0$ on ∂S and

$$S' \subset S,\ S' \neq S \Longrightarrow \beta_{S'} > \beta_S.$$

2.2 Non-existence

Proof of Theorem 1. We put

$$\theta = \frac{\beta_q}{\beta_S} \text{ and } \eta = \phi_S^\theta.$$

Then $\theta \geq 1$ and

$$\nabla'\eta = \theta\phi_S^{\theta-1}\nabla'\phi_S,$$

$$\beta_q^2\eta^2 + |\nabla'\eta|^2 = \theta^2\phi_S^{2(\theta-1)}(\beta_S^2\phi_S^2 + |\nabla'\phi_S|^2),$$

$$(\beta_q^2\eta^2 + |\nabla'\eta|^2)^{(p-2)/2} = \theta^{p-2}\phi_S^{(p-2)(\theta-1)}(\beta_S^2\phi_S^2 + |\nabla'\phi_S|^2)^{(p-2)/2},$$

$$\begin{aligned}\nabla'.(\beta_q^2\eta^2 + |\nabla'\eta|^2)^{(p-2)/2}\nabla'\eta = {}& \theta^{p-1}\phi_S^{(p-1)(\theta-1)}\nabla'.(\beta_S^2\phi_S^2 + |\nabla'\phi_S|^2)^{(p-2)/2}\nabla'\phi_S \\ & + (p-1)(\theta-1)\theta^{p-2}\phi_S^{(p-1)(\theta-1)-1}(\beta_S^2\phi_S^2 + |\nabla'\phi_S|^2)^{(p-2)/2}|\nabla'\phi_S|^2\end{aligned}$$

Using (2.4) with $\phi = \phi_S$, we derive

$$\mathcal{T}(\eta) = -(p-1)\theta^{p-1}(\theta-1)\phi_S^{(p-1)(\theta-1)-1}(\beta_S^2\phi_S^2 + |\nabla'\phi_S|^2)^{p/2} \text{ in } S. \tag{2.6}$$

Because ω is a nonnegative nontrivial solution of (2.3), it is nonpositive in S. Furthermore $\partial\omega/\partial\nu < 0$ on ∂S. Therefore we can choose ϕ_S as the maximal positive solution of (2.4) such that $\eta \leq \omega$. If $\theta > 1$ there exists $\sigma^* \in S$ such that

$$\omega(\sigma^*) = \eta(\sigma^*) > 0 \;\text{ and }\; \omega(\sigma) \geq \eta(\sigma) \quad \forall \sigma \in \bar{S}. \tag{2.7}$$

If $\theta = 1$, the graphs of ω and η could be tangent only on ∂S. This means that either (2.7) holds, or there exists $\bar{\sigma} \in \partial S$ such that

$$\partial\omega(\bar{\sigma})/\partial\nu = \partial\eta(\bar{\sigma})/\partial\nu < 0 \;\text{ and }\; \omega(\sigma) < \eta(\sigma) \quad \forall \sigma \in S. \tag{2.8}$$

Let $\psi = \omega - \eta$ and we first consider the case where (2.7) holds. Let $g = (g_{ij})$ be the metric tensor on S^{N-1}. We recall the following expressions in local coordinates σ_j around σ^*,

$$|\nabla'\varphi|^2 = \sum_{j,k} g^{jk} \frac{\partial\varphi}{\partial\sigma_j}\frac{\partial\varphi}{\partial\sigma_k},$$

for any $\varphi \in C^1(S)$, and

$$\nabla'.X = \frac{1}{\sqrt{|g|}} \sum_{\ell} \frac{\partial}{\partial\sigma_\ell}\left(\sqrt{|g|}X^\ell\right) = \frac{1}{\sqrt{|g|}} \sum_{\ell,i} \frac{\partial}{\partial\sigma_\ell}\left(\sqrt{|g|}g^{\ell i}X_i\right),$$

for any vector field $X \in C^1(TS^{N-1})$, if we lower the indices by setting $X^\ell = \sum_i g^{\ell i}X_i$. We derive from the mean value theorem

$$\begin{aligned}(\beta_q^2\omega^2 + |\nabla'\omega|^2)^{(p-2)/2}\frac{\partial\omega}{\partial\sigma_i} &- (\beta_q^2\eta^2 + |\nabla'\eta|^2)^{(p-2)/2}\frac{\partial\eta}{\partial\sigma_i} \\ &= \sum_j \alpha_j^i \frac{\partial(\omega-\eta)}{\partial\sigma_j} + b^i(\omega-\eta),\end{aligned}$$

where

$$\begin{aligned}b^i = (p-2)\left(\beta_q^2(\eta + t(\omega-\eta))^2 + |\nabla'(\eta + t(\omega-\eta))|^2\right)^{(p-4)/2} \\ \times(\eta + t(\omega-\eta))\frac{\partial(\eta + t(\omega-\eta))}{\partial\sigma_i},\end{aligned}$$

and

$$\begin{aligned}\alpha_j^i = (p-2)\left(\beta_q^2(\eta + t(\omega-\eta))^2 + |\nabla'(\eta + t(\omega-\eta))|^2\right)^{(p-4)/2} \\ \times\frac{\partial(\eta + t(\omega-\eta))}{\partial\sigma_i}\sum_k g^{jk}\frac{\partial(\eta + t(\omega-\eta))}{\partial\sigma_k} \\ +\delta_i^j\left(\beta_q^2(\eta + t(\omega-\eta))^2 + |\nabla'(\eta + t(\omega-\eta))|^2\right)^{(p-2)/2}.\end{aligned}$$

Since the graph of η and ω are tangent at σ^*,

$$\eta(\sigma^*) = \omega(\sigma^*) = P_0 > 0 \quad \text{and } \nabla'\eta(\sigma^*) = \nabla'\omega(\sigma^*) = Q.$$

Thus

$$b^i(\sigma^*) = (p-2)\left(\beta_q^2 P_0^2 + |Q|^2\right)^{(p-4)/2} P_0 Q_i,$$

and

$$\alpha_j^i(\sigma^*) = \left(\beta_q^2 P_0^2 + |Q|^2\right)^{(p-4)/2} \left(\delta_i^j(\beta_q^2 P_0^2 + |Q|^2) + (p-2)Q_i \sum_k g^{jk} Q_k\right).$$

Now

$$\begin{aligned}
\mathcal{T}(\omega) - \mathcal{T}(\eta) &= \omega^q + (p-1)\theta^{p-1}(\theta-1)\phi_S^{(p-1)(\theta-1)-1}(\beta_S^2\phi_S^2 + |\nabla'\phi_S|^2)^{p/2} \\
&= \frac{-1}{\sqrt{|g|}} \sum_{\ell,i} \frac{\partial}{\partial\sigma_\ell}\left[\sqrt{|g|} g^{\ell i}\left((\beta_q^2\omega^2 + |\nabla'\omega|^2)^{\frac{p}{2}-1}\frac{\partial\omega}{\partial\sigma_i} - (\beta_q^2\eta^2 + |\nabla'\eta|^2)^{\frac{p}{2}-1}\frac{\partial\eta}{\partial\sigma_i}\right)\right] \\
&\qquad -\lambda_{q,p}\left((\beta_q^2\omega^2 + |\nabla'\omega|^2)^{\frac{p}{2}-1}\omega - (\beta_q^2\eta^2 + |\nabla'\eta|^2)^{\frac{p}{2}-1}\eta\right), \\
&= -\frac{1}{\sqrt{|g|}}\sum_{\ell,i}\frac{\partial}{\partial\sigma_\ell}\left[\sqrt{|g|}g^{\ell i}\left(\sum_j \alpha_j^i \frac{\partial(\omega-\eta)}{\partial\sigma_j} + b^i(\omega-\eta)\right)\right] + \sum_i C_i \frac{\partial(\omega-\eta)}{\partial\sigma_i} \\
&= -\frac{1}{\sqrt{|g|}}\sum_{\ell,j}\frac{\partial}{\partial\sigma_\ell}\left[a_j^\ell\frac{\partial(\omega-\eta)}{\partial\sigma_j}\right] + \sum_i C_i\frac{\partial(\omega-\eta)}{\partial\sigma_i} \qquad ,
\end{aligned}$$

where the C_i are continuous functions and

$$a_j^\ell = \sqrt{|g|}\sum_i g^{\ell i}\alpha_j^i.$$

The matrix $\left(\alpha_j^i(\sigma_0)\right)$ is symmetric, definite and positive since it is the Hessian of the strictly convex function

$$X = (X_1, \dots, X_{n-1}) \mapsto \frac{1}{p}\left(P_0^2 + |X|^2\right)^{p/2} = \frac{1}{p}\left(P_0^2 + \sum_{j,k} g^{jk}X_jX_k\right)^{p/2}.$$

Therefore $\left(\alpha_j^i\right)$ has the same property in some neighborhood of σ^*, and the same holds true with $\left(a_j^\ell\right)$. Finally the function $\psi = \omega - \eta$ is nonnegative, vanishes at σ^* and satifies

$$-\frac{1}{\sqrt{|g|}}\sum_{\ell,j}\frac{\partial}{\partial\sigma_\ell}\left[a_j^\ell\frac{\partial\psi}{\partial\sigma_j}\right] + \sum_i C_i\frac{\partial\psi}{\partial\sigma_i} \geq 0. \tag{2.9}$$

Then $\psi = 0$ in a neighborhood of S. Since S is connected, ψ is identically 0 which a contradiction.

If (2.8) holds, then $\theta = 1$ and the graphs of η and ω are tangent at $\bar\sigma$. Proceeding as above and using the fact that $\partial\eta/\partial\nu$ exists and never vanishes on the boundary, we see that $\psi = \eta - \omega$ satisfies (2.9) with a strongly elliptic operator in a neighborhood $\mathcal{N}$ of $\bar\sigma$. Moreover $\psi > 0$ in $\mathcal{N}$, $\psi(\bar\sigma) = 0$ and $\partial\psi/\partial\nu(\bar\sigma) = 0$. This is a contradiction, which ends the proof. $\square$

Remark. If $p = 2$, the proof of non-existence is straightforward by multiplying the equation in ω by the first eigenfunction ϕ_S and get

$$\int_S ((\lambda_S - \lambda_{q,2})\omega - \omega^q)\, \phi_S d\sigma = 0,$$

a contradiction since $\lambda_S \leq \lambda_{q,2}$.

2.3 Existence results

Let us consider the case $q = q_c = (N(p-1)+p)/(N-p)$ $(N > p > 1)$, and let S be any smooth subdomain of S^{N-1}. The research of solution of (1.1) under the form (1.2) vanishing on ∂C_S leads to

$$\begin{cases} -\nabla'.\left(\left(\beta_{q_c}^2\omega^2 + |\nabla'\omega|^2\right)^{p/2-1}\nabla'\omega\right) - |\omega|^{q_c-1}\omega = -\beta_{q_c}^2\left(\beta_{q_c}^2\omega^2 + |\nabla'\omega|^2\right)^{p/2-1}\omega \quad \text{in } S \\ \omega = 0 \quad \text{in } \partial S, \end{cases} \tag{2.10}$$

where $\beta_{q_c} = N/p - 1$. This equation is the Euler-Lagrange variation of the functional J defined on $W_0^{1,p}(S)$ by

$$J(\psi) = \int_S \left(\frac{1}{p}\left(\beta_{q_c}^2\psi^2 + |\nabla'\psi|^2\right)^{p/2} - \frac{1}{q_c+1}|\psi|^{q_c+1}\right) d\sigma. \tag{2.11}$$

Theorem 2.1 *Problem (2.10) admits a positive solution.*

Proof. Clearly the functional is well defined on $W_0^{1,p}(S)$ since q_c is smaller than the Sobolev exponent p^*_{N-1} for $W^{1,p}$ in dimension N-1. For any $\psi \in W_0^{1,p}(S)$, $\lim_{t\to\infty} J(t\psi) = -\infty$. Furthermore there exist $\delta > 0$ and $\epsilon > 0$ such that $J(\psi) \geq \epsilon$ for any $\psi \in W_0^{1,p}(S)$ such that $\|\psi\|_{W^{1,p}} = \delta$. Assume now that $\{\psi_n\}$ is a sequence of $W_0^{1,p}(S)$ such that $J(\psi_n) \to \alpha$ and $\|DJ(\psi_n)\|_{W^{-1,p'}} \to 0$ as $n \to \infty$. Then

$$-\nabla'.\left(\left(\beta_{q_c}^2\psi_n^2 + |\nabla'\psi_n|^2\right)^{p/2-1}\nabla'\psi_n\right) - |\psi_n|^{q_c-1}\psi_n + \beta_{q_c}^2\left(\beta_{q_c}^2\psi_n^2 + |\nabla'\psi_n|^2\right)^{p/2-1}\psi_n = \epsilon_n \to 0.$$

Then

$$\int_S \left(\left(\beta_{q_c}^2\psi_n^2 + |\nabla'\psi_n|^2\right)^{p/2} - |\psi_n|^{q_c+1}\right) d\sigma = \langle \epsilon_n, \psi_n \rangle.$$

Since $J(\psi_n) \to \alpha$ it follows

$$\int_S \left(\beta_{q_c}^2\psi_n^2 + |\nabla'\psi_n|^2\right)^{p/2} d\sigma \to p(q_c+1)\alpha/(q_c+1-p).$$

Therefore $\{\psi_n\}$ remains bounded in $L^{q_c+1}(S)$, and relatively compact in $L^r(S)$, for any $1 < r < q_c + 1$. Multiplying the equation $DJ(\psi_n) - \epsilon_n$ by $T_{k,\theta}(\psi_n)$ where $\theta \in (1, (p^*_{N-1} - 1)/q_c)$,

$k > 0$ and $T_{k,\theta}(r) = \operatorname{sgn}\min\{|r|, k\}$ and using standard bootstrap arguments yields to the boundedness of $\{\psi_n\}$ in $L^\infty(S)$. Combining this fact with the compactness of in $L^r(S)$, we derive the compactness in any any L^s, for $s < \infty$. Therefore $\{\psi_n\}$ is relatively compact in $W_0^{1,p}(S)$. This means that J satisfies the Palais-Smale condition. $\square$

3 The 2-dim dynamical system

3.1 Extension of the data

We study more generally the existence of 2π-periodic solutions of equation

$$\frac{d}{d\sigma}\left[\left(\beta^2\omega^2 + \left(\frac{d\omega}{d\sigma}\right)^2\right)^{p/2-1}\frac{d\omega}{d\sigma}\right] + \lambda\left[\beta^2\omega^2 + \left(\frac{d\omega}{d\sigma}\right)^2\right]^{p/2-1}\omega + g(\omega) - c|\omega|^{p-2}\omega = 0, \tag{3.1}$$

where λ, β, c are real arbitrary parameters, with $\beta > 0$, and $g \in C^0(\mathbb{R}) \cap C^1(\mathbb{R}\setminus\{0\})$, g is odd,

$$\lim_{s\to 0+} g(s)/s^q = 1, \qquad \lim_{s\to\infty} g(s)/s^{p-1} = \infty, \qquad \frac{d}{ds}(g(s)/|s|^{p-1}) > 0 \text{ on } (0,\infty), \tag{3.2}$$

with $q > p - 1 \geq 0$. In fact we can easily reduce the problem to a simpler form, and particularly in the case $p = 1$, where the equation has a remarkable homogeneity property. The next statement is straightforward.

Lemma 3.1 *(i) Let $p > 1$. Setting*

$$\tau = \beta\sigma, \ \text{ and } \omega(\sigma) = \beta^{p/(q+1-p)} w(\tau); \tag{3.3}$$

equation (3.1) takes the form, where $w' = dw/d\tau$:

$$\frac{d}{d\tau}\left(\left(w^2 + w'^2\right)^{p/2-1} w'\right) - b\left(w^2 + w'^2\right)^{p/2-1} w + f(w) - d|w|^{p-2}w = 0, \tag{3.4}$$

where

$$b = \frac{-\lambda}{\beta^2}, \qquad d = \frac{c}{\beta^p}, \qquad f(s) = \beta^{-pq/(q+1-p)} g(\beta^{p/(q+1-p)}s), \tag{3.5}$$

thus f satisfies the same assumptions (3.2) as g.
(ii) Let $p = 1$. At any point where $\omega(\sigma) \neq 0$, setting

$$\tau = \beta q\,\sigma, \quad \text{and } \omega(\sigma) = (\beta q)^{1/q}|w(\tau)|^{1/q-1} w(\tau), \tag{3.6}$$

then w satisfies an equation of type (3.4) with

$$b = -\lambda/\beta^2 q, \qquad d = c/\beta q, \qquad f_1(s) = \beta^{-q} g((\beta q s)^{1/q}), \tag{3.7}$$

and f_1 satisfies the assumptions (3.2) with $q = 1$:

$$\lim_{s\to 0+} f_1(s)/s = 1, \qquad \lim_{s\to\infty} f_1(s) = \infty, \qquad f_1'(s) > 0 \text{ on } (0,\infty), \tag{3.8}$$

From above, the changes of variables (3.3) and (3.6) reduce the problem to study the existence of periodic solutions of equation (3.4) and the range of values of the period function, for any $q > p - 1$ if $p > 1$, and for $q > 0$ if $p = 1$.

3.2 Reduction to dynamical systems

Assume $p \geq 1$. Equation (3.4) can be re-written as the system,

$$\begin{cases} w' = F(w,y) = y \\ y' = G(w,y) = \dfrac{bw^3 + (b+2-p)w\,y^2 - (f(w) - d|w|^{p-2}w)(w^2+y^2)^{2-p/2}}{w^2+(p-1)y^2}. \end{cases} \tag{3.9}$$

We denote by h the odd function defined on $\mathbb{R}$ by

$$h(s) = \begin{cases} f(s)/\left|s\right|^{p-2} s & \text{if } s \neq 0 \\ 0 & \text{if } s = 0. \end{cases} \tag{3.10}$$

If $b+d \leq 0$, it has no non-trivial stationary point, while if $b+d > 0$, it admits the two stationary points $\pm P_0$, with

$$P_0 = (a, 0), \qquad a = h^{-1}(b+d).$$

Furthermore P_0 is a center since the linearized system at P_0 is given by the matrix $\begin{pmatrix} 0 & 1 \\ -ah'(a) & 0 \end{pmatrix}$.

System (3.9) is singular at $(0,0)$, and, *a priori*, along the line $w = 0$ if $p < 2$, and $q < 1$ or $d \neq 0$, or if $p = 1$. In fact for $p > 1$ it is not singular at the points $(0,\sigma)$ for $\sigma \neq 0$. Indeed the problem $w'' = G(w,w')$ with initial data $w(0) = 0, w'(0) = \sigma$ has a unique local solution: indeed near $(0,\sigma)$, G is continuous with respect to w and C^1 with respect to y, thus w is C^2; locally t can be expressed in terms of w, and setting $w'(t) = p(w)$, p is C^1 near 0, $p(0) = 1$ and $dp/dw = G(w,p)/p$, and $J(w,p) = G(w,p)/p$ is C^1 with respect to p and continuous with respect to w, thus one gets local uniqueness of p; and then the local uniqueness of problem $w'(t) = p(w(t))$, $w(0) = 1$, since p is of class C^1.

The phase plane of system (3.9) is invariant by symmetries with respect to the two axes of coordinates, because F is even with respect to w and odd with respect to y, and G is odd with respect to w and even with respect to y. Thus from now we can restrict the study to the first quadrant

$$\overline{\mathcal{Q}} \backslash \{(0,0)\}, \quad \text{where } \mathcal{Q} = (0,\infty) \times (0,\infty);$$

in particular $w \geq 0$. In case $p > 1$, due to the symmetries, any trajectory which meets the two axes in finite times $\tau, \tau + T$ is a closed orbit of period $4T$.

Remark. It is sometimes useful to introduce the slope $\xi = w'/w$, (or a function of the slope) as a new variable. This was first used for $p > 1$ in [11] for the homogeneous problem

$$\frac{d}{d\tau}\left(\left(w^2 + w'^2\right)^{p/2-1} w'\right) - b\left(w^2 + w'^2\right)^{p/2-1} w = 0;$$

in that case it be written under the form

$$\frac{d}{d\tau}\left(\left(1+\xi^2\right)^{p/2-1}\xi\right) = -((p-1)\xi^2 - b)\left(1+\xi^2\right)^{p/2-1},$$

for $w > 0$, and this equation is completely integrable in terms of $u = \left(1+\xi^2\right)^{p/2-1}\xi$.

We can transform (3.9) by using polar coordinates in $\mathcal{Q}$

$$(w, y) = (\rho\cos\theta, \rho\sin\theta), \qquad \rho > 0, \theta \in (0, \pi/2).$$

It gives

$$\theta' = \frac{b - (p-1)\tan^2\theta + (d - h(\rho\cos\theta))\cos^{p-2}\theta}{1 + (p-1)\tan^2\theta}, \qquad \rho' = \rho(1+\theta')\tan\theta. \tag{3.11}$$

Equivalently, we introduce the slope $\xi = \tan\theta \in (0, \infty)$ instead of the angle, and set

$$u = \phi(\xi) = \cos^{1-p}\theta\sin\theta, \qquad \phi(\xi) = (1+\xi^2)^{(p-2)/2}\xi,$$

the function ϕ is strictly increasing, from $(0, \infty)$ into $(0, \infty)$ when $p > 1$, from $(0, \infty)$ into $(0, 1)$ when $p = 1$. Indeed $\phi'(\xi) = (1+\xi^2)^{(p-4)/2}(1 + (p-1)\xi^2)$. Defining

$$\varphi = \phi^{-1}, \quad \text{and} \quad E(\xi) = \left((p-1)\xi^2 - b\right)(1+\xi^2)^{p/2-1}, \tag{3.12}$$

we obtain

$$\begin{cases} w' = w\varphi(u), \\ u' = -E(\varphi(u)) - h(w) + d. \end{cases} \tag{3.13}$$

This system is still singular on the line $w = 0$ if $h \notin C^1([0, \infty))$ near 0. In the sequel we set

$$\Psi(u) = \int_0^u \varphi(s)ds. \tag{3.14}$$

Noticing that

$$E'(\xi) = \left(p(p-1)\xi^2 + 2(p-1) - (p-2)b\right)(1+\xi^2)^{(p-2)/2}\xi, \tag{3.15}$$

we derive that E is increasing on $(0, \infty)$ when $(p-2)b \leq 2(p-1)$. When $(p-2)b > 2(p-1)$, E is decreasing on $(0, \eta)$ and then increasing, where

$$p(p-1)\eta^2 = (p-2)b - 2(p-1), \tag{3.16}$$

and

$$\min E = E(\eta) = -\frac{2}{p-2}\left(\frac{(p-2)(b+p-1)}{p(p-1)}\right)^{p/2}. \tag{3.17}$$

In the case of initial problem (1.11), E is increasing.

Remark. If $p > 1$, system (3.9) is singular at $(0, 0)$. If we replace the assumption $\lim_{s\to 0+} f(s)/s^q = 1$, by the stronger one

$$\lim_{s\to 0+} f'(s)/s^{q-1} = q, \tag{3.18}$$

we can transform system (3.13) in $(0,\infty)\times\mathbb{R}$ in a system of the same type, but without singularity, by performing the substitution

$$v = w^{q+1-p}.$$

Then

$$v' = (q+1-p)v\varphi(u), \qquad u' = -E(\varphi(u)) - \tilde{h}(v) + d. \tag{3.19}$$

where $v \mapsto \tilde{h}(v) = h(v^{1/(q+1-p)}) \in C^1([0,1)$. In particular, if $f(w) = |w|^{q-1}w$, we find

$$v' = (q+1-p)v\varphi(u), \qquad u' = -E(\varphi(u)) - v + d. \tag{3.20}$$

Remark. In the case $f(w) = |w|^{q-1}w$, we can differentiate the equation relative to u' and obtain a second order equation that satisfies u,

$$u'' = B(\varphi(u))u' + (q+1-p)(E(\varphi(u)) - d)\varphi(u) \tag{3.21}$$

where E is given above, and

$$B(\xi) = \frac{(p-2)b + q - 3(p-1) + (q+1-2p)(p-1)\xi^2}{1+(p-1)\xi^2}\xi. \tag{3.22}$$

Notice that equation (3.21) has no singularity for $p > 1$.

4 The case $p > 1$

4.1 Existence of a first integral

A natural question is to see if equation (3.4) admits a variational structure.

When $p = 2$, it is the case, for any reals b, d : indeed (3.4) takes the form

$$w'' - (b+d)w + f(w) = 0,$$

hence denoting $\mathcal{F}(w) = \displaystyle\int_0^w f(s)ds$, it is the Euler equation of the functional

$$\mathcal{H}_2(w, w') = \frac{w'^2}{2} + (b+d)\frac{w^2}{2} - \mathcal{F}(w)$$

and thus it has a first integral $w'^2 = (b+d)w^2 - 2\mathcal{F}(w) + C$.

When $p \neq 2, p > 1$, we find a first integral only in the case $b = 1$; in this case equation (3.4) is the Euler equation of the functional

$$\mathcal{H}(w, w') = \frac{\left(w^2 + w'^2\right)^{p/2}}{p} + d\frac{|w|^p}{p} - \mathcal{F}(w),$$

from which follows the Painlevé integral:

$$\frac{1}{p}\left(w^2 + w'^2\right)^{p/2-1}\left((p-1)w'^2 - w^2\right) = d\frac{|w|^p}{p} - \mathcal{F}(w) - K. \tag{4.1}$$

Using the function E introduced at (3.12), then (4.1) is equivalent for $w > 0$ to

$$E\left(\frac{w'}{w}\right) = E(\varphi(u)) = d - p\frac{K + \mathcal{F}(w)}{w^p} \tag{4.2}$$

and here E is increasing on $(0, \infty)$ from $-b = -1$ to $+\infty$.

Therefore, in the general case, we cannot use a first integral for studying the periodicity properties of the solutions, while it was the main tool in [2] for $p = 2$. We are lead to use phase plane techniques. For the initial problem (1.11), the value $b = 1$ corresponds to the case $p < 2$ and $q = (3p - 2)/(2 - p))$.

4.2 Description of the solutions

Next we describe in more details the trajectories of system (3.9) in the phase plane (w, y). Recall that the system can be singular on the axis $w = 0$.

Proposition 4.1 *Assume $p > 1$. Then all the orbits of system (3.9) are bounded. Any trajectory $\mathcal{T}_{[P]}$ issued of any point P in $\mathcal{Q}$ is either a closed orbit surrounding $(0, 0)$, or (only if $b + d > 0$) a closed orbit surrounding P_0 but not $(0, 0)$, or an homoclinic orbit, starting from $(0, 0)$ with the slope m such that $E(m) = d$, ending at $(0, 0)$ with the slope $-m$ and with $\mathbb{R}$ for interval of existence.*

Proof. First look at the vector field on the boundary of $\mathcal{Q}$. At any point $(0, \sigma)$ with $\sigma > 0$, it is given by $(\sigma, 0)$, thus it is transverse and inward. At any point $(\bar{w}, 0)$ with $\bar{w} > 0$, it is given by $(0, \bar{w}(b + d - h(\bar{w}))$. Thus it is transverse and outward whenever $b + d \leq 0$ or $b + d > 0$ and $\bar{w} > a$, and inward whenever $b + d > 0$ and $\bar{w} < a$.

Consider any solution (w, y) of the system, such that $P = (w(0), y(0)) \in \mathcal{Q}$, and let (τ_1, τ_2) be its maximal interval existence in $\mathcal{Q}$. At each point τ where $u'(\tau) = 0$, and $u > 0$, $u''(\tau) = -h'(w)w\varphi(u) < 0$ from (3.13). Thus if τ exists, it is unique, and it is a maximum for u.

Since $w' = y > 0$, then w has a limit $\ell_2 \in (0, \infty]$ at τ_2 and $\ell_1 \in [0, \infty)$ at τ_1. And u is strictly monotonous near τ_1, τ_2, thus it has limits $u_1, u_2 \in [0, \infty]$, in other words θ has limits $\theta_1, \theta_2 \in [0, \pi/2]$

(i) Let us go forward in time. On any interval where u is increasing, one has $E(\varphi(u)) \leq d$, thus u is bounded. Then u_2 is finite. If $\ell_2 = \infty$, then θ' tends to $-\infty$, thus ρ is decreasing thus it is bounded, which is contradictory; thus ℓ_2 is finite. If $u_2 > 0$ then $(\ell_2, \ell_2\varphi(u_2))$ is stationary, which is impossible. Thus u is decreasing to 0, and the trajectory converges to $(\ell_2, 0)$. If $b + d > 0$ and $\ell_2 = a$, then u' tends to 0 from (3.13), and

$$u'' = -(E \circ \varphi)'(u)u' - h'(w)w\varphi(u) = -h'(a)a\varphi(u)(1 + o(1))$$

thus $u'' < 0$ near τ_2, which is impossible. Then either $b + d \leq 0$, or $b + d > 0$ and $\bar{w} > a$, and τ_2 is finite, the trajectory leaves $\mathcal{Q}$ transversally at τ_2.

(ii) Next let us go backward in time.

• Suppose that $u_1 = 0$, then the trajectory converges to $(\ell_1, 0)$; then necessarily $b+d > 0$ and $\ell_1 \leq a$, and $\ell_1 < a$ as above . The trajectory enters $\mathcal{Q}$ transversally at τ_2, and from the symmetries it is a closed orbit surrounding only the stationnary point P_0.

• Suppose that $u_1 = \infty$, that means θ tends to $\pi/2$. Then from (3.11), θ' tends to 1, thus τ_1 is finite and $\pi/2 - \theta = (\tau - \tau_1)(1 + o(1))$, $\tan\theta = (\tau - \tau_1)^{-1}(1 + o(1))$, and

$$(p-1)(\tau - \tau_1)^{-1}\frac{\rho'}{\rho} = (b + 1 + (d - h(\rho\cos\theta))\cos^{p-2}\theta)(1 + o(1)$$

If $p \geq 2$, then $\rho'/\rho = O((\tau - \tau_1))$; if $p < 2$ then $\rho'/\rho = O((\tau - \tau_1)^{p-1})$. In any case, $\ln\rho$ case is integrable, thus ρ has a finite limit $\bar{y} > 0$. Then the trajectory enters $\mathcal{Q}$ transversally at τ_1 and from the symmeries it is a closed orbit surrounding $(0,0)$. From the considerations in § 3-2, for any $\bar{y} > 0$ there exist such an orbit, and it is unique. Moreover in $\mathcal{Q}$ the slope $\xi = \varphi(u)$ is decreasing from ∞ to 0; indeed it decreases near τ_1 and τ_2 and can only have a maximal point.

• Suppose that $0 < u_1 < \infty$. If $\ell_1 > 0$, then $(\ell_1, \ell_1\varphi(u_1))$ is stationary, which is impossible. Thus (y, w) converges to $(0,0)$. And w'/w tends to $\varphi(u_1)$, thus $\tau_1 = -\infty$. And u' converges to $d - E(\varphi(u_1))$, thus $\tan\theta = \varphi(u)$ has a limit $m \geq 0$ such that $E(m) = d$. From the symmetries the trajectory is homoclinic and the solution w is defined on $\mathbb{R}$. □

The next precises the behaviour of solutions according to the sign of $b + d$.

Theorem 4.2 *Consider system (3.9) with* $p > 1$.

(i) Assume $b+d > 0$. *Then there exists a unique homoclinic trajectory* $\mathcal{H}$ *starting from* $(0,0)$ *in* $\mathcal{Q}$ *with the slope* $m_d = E^{-1}(d)$ *(* $m_0 = \sqrt{b/(p-1)}$ *if* $d = 0$*), and ending at* $(0,0)$ *with the slope* $-m_d$*, and surrounding* P_0*. Up to the stationary points, the other orbits are closed, and either they surround only one of the points* P_0 *or* $-P_0$*, in the domain delimitated by* $\mathcal{H}$*, corresponding to solutions* w *of constant sign, or they are exterior to* $\pm\mathcal{H}$ *and surround* $(0,0)$ *and* $\pm P_0$*, corresponding to sign changing solutions* w*.*

(ii) Assume $b + d \leq 0$*. Then*

• *if* $(p-2)b \leq 2(p-1)$*, or* $[(p-2)b > 2(p-1)$ *and* $d < E(\eta)]$*, there is no homoclinic trajectory.*

• *if* $[(p-2)b > 2(p-1)$ *and* $E(\eta) < d \leq -b]$*; then denoting by* $m_{1,d} < m_{2,d}$ *the two roots of equation* $E(m) = d$*, there exists an infinity of homoclinic trajectories* $\mathcal{H}_1$ *starting from* $(0,0)$ *in* $\mathcal{Q}$ *with the slope* m_1 *and ending at* $(0,0)$ *with the slope* $-m_{1,d}$*, and a unique homoclinic trajectory* $\mathcal{H}_2$ *starting from* $(0,0)$ *in* $\mathcal{Q}$ *with the slope* m_2 *and ending at* $(0,0)$ *with the slope* $-m_2$*.*

Proof. (i) **Case** $b+d > 0$. Then the equation $E(m) = d$ has a unique solution $m = E^{-1}(d)$; and w'/w tends to m; thus the trajectory starts from $(0,0)$ with a slope m.Then for any $P \in \mathcal{Q}$, the trajectory $\mathcal{T}_{[P]}$ passing through P meets the axis $y = 0$ after P at some point $(\mu, 0)$ with $\mu > a$. Let

$$\mathcal{U} = \left\{P \in \mathcal{Q} : \mathcal{T}_{[P]} \cap \{(0,\sigma) : \sigma > 0\} \neq \emptyset\right\}, \qquad \mathcal{V} = \left\{P \in \mathcal{Q} : \mathcal{T}_{[P]} \cap \{(\mu, 0) : 0 < \mu < a\} \neq \emptyset\right\}.$$

Then either $P \in \mathcal{U}$ and the trajectory is a closed orbit surrounding $(0,0)$ and $\pm P_0$, and in $\mathcal{Q}$. Or $P \in \mathcal{V}$ and the trajectory is a closed orbit surrounding only P_0. Or $\mathcal{T}_{[P]}$ is an homoclinic orbit $\mathcal{H}$ starting from (0,0) with the slope m, where m is the unique solution of equation $E(m) = d$ (, such that $m > \eta$ if E is not monotone, see (3.15). Now $\mathcal{U}$ and $\mathcal{V}$ are open, since the vector field on the axes is transverse, thus $\mathcal{U} \cup \mathcal{V} \neq \mathcal{Q}$. This shows the existence of such an orbit $\mathcal{H}$.

(ii) **Case** $b + d \leq 0$.

• Either $b + d < 0$ and E is increasing, or E has a minimum at η and $d < E(\eta)$. Then equation $E(m) = d$ has no solution: there is no homoclinic orbit. Or E is increasing and $b + d = 0$; then $E(\varphi(u)) > -b = d$, thus $u' < 0$, thus u cannot tend to 0, the same conclusion holds.

• 0r E has a minimum at η and $E(\eta) < d \leq -b$; then the equation $E(m) = d$ has two roots m_1, m_2 such that $0 \leq m_1 < \eta < m_2 \leq m_b$, where $E(m_b) = -b$. Consider again the set $\mathcal{U}$. Any trajectory $\mathcal{T}_{[P]}$ such that $P \in \mathcal{U}$ satisfies $u' < 0$, that means $h(w) > d - E(\varphi(u))$ and u describes $(0, \infty)$, thus there exists τ such that $\varphi(u)(\tau) = \eta$, thus $h(w(\tau)) > d - E(\eta)$ and $y(\tau) = \eta w(\tau)$. Next consider any trajectory $\mathcal{T}_{[\tilde{P}]}$ going through $\tilde{P} = (\tilde{w}, \eta\tilde{w})$ such that $h(\tilde{w}) \leq d - E(\eta)$ at time 0. It cannot be a trajectory of the preceeding type, thus $(y, w) \to (0, 0)$ as $\tau \to \tau_1$, and θ tends to θ_1, with $\tan\theta_1 = m_1$ or m_2; moreover $u'(0) \geq 0$, and $u' < 0$ near τ_2, thus there exists a unique $\tau \geq 0$ such that $u'(\tau) = 0$; then $u' > 0$ in (τ_1, τ), thus $\tan\theta_1 < \eta$, thus $\tan\theta_1 = m_1$.Thus there exist an infinity of such trajectories $\mathcal{H}_1$, starting with the slope m_1. Next fix one trajectory $\mathcal{T}_{[\tilde{P}_0]}$ such that $h(\tilde{w}_0) \leq d - E(\eta)$. Let $\mathcal{R}$ be the subdomain of $\mathcal{Q}$ delimitated by $\mathcal{T}_{[\tilde{P}_0]}$ and $\mathcal{T}_{[(0,1)]}$. Let

$$\mathcal{V} = \left\{P \in \mathcal{R} : \mathcal{T}_{[P]} \cap \{(w, \eta w) : 0 < w < \tilde{w}_0\} \neq \emptyset\right\}.$$

Then also $\mathcal{V}$ is open. Indeed the intersection with the line $y = \eta w$ for $w < \tilde{w}_0$ is transverse: at the intersection point, $h(\tilde{w}) < d - E(\eta)$, thus $u' > 0$, and $y/w = \varphi(u) = \eta$, and

$$\frac{y'}{y} = \varphi(u) + \frac{\varphi'(u)}{\varphi(u)}u' > \eta = \frac{w'}{w}$$

Then $(\mathcal{U} \cap \mathcal{R}) \cup \mathcal{V} \neq \mathcal{R}$. Then there exists at least a trajectory $\mathcal{H}_{1,*}$ starting from $(0, 0)$ with the slope m_2.

(iii) **Uniqueness of** $\mathcal{H}$ and $\mathcal{H}_2$. Let $m = m_0$ or $m_{2,d}$. Suppose that system (3.9) has two solutions (w_1, y_1), (w_2, y_2) near $-\infty$, such that $w_i > 0$ and w_i tends to 0 and y_i/w_i tend to m. Then the system (3.13) has two local solutions (w_1, u_1), (w_2, u_2) such that $\varphi(u_i)$ tends to m at $-\infty$. Then $w_i' > 0$ locally, thus one can express u_i as a function of w_i. Then at the same point w,

$$w\frac{d(\Psi(u_i)}{dw} = w\varphi(u_i)\frac{du_i}{dw} = -E(\varphi(u_i)) - h(w) + d,$$

$$w\frac{d(\Psi(u_2) - \Psi(u_1))}{dw} = -(E(\varphi(u_2)) - E(\varphi(u_1)) = E'(\varphi(u^*))\varphi'(u^*)(u_2 - u_1)$$

for some u^* between u_1 and u_2, and $E'(\varphi(u^*)) = E'(m)(1 + o(1))$; and $E'(m) > 0$. Then for small w

$$\frac{d(\Psi(u_2) - \Psi(u_1))}{dw}(\Psi(u_2) - \Psi(u_1)) < 0$$

thus $(\Psi(u_2)-\Psi(u_1))^2$ is decreasing; and its limit at 0 is 0, thus $\Psi(u_2)=\Psi(u_1)$, thus $u_2\equiv u_1$ near $-\infty$ then from (3.13), $h(w_1)=h(w_2)$, and h is injective, thus $w_1\equiv w_2$ near $-\infty$. The global uniqueness follows, since the system is regular except at $(0,0)$. And all the trajectories are described.

Remark. Under the assumption (3.18), existence and uniqueness of $\mathcal{H}$ and $\mathcal{H}_2$ can be obtained more directly whenever $d\neq E(\eta)$. Indeed the system (3.19) relative to (v,u) is regular, with stationary points $(0,0)$, $(0,\pm\varphi^{-1}(m))$, where $m=m_0,m_1$ or m_2 and also $(\pm a,0)$ if $b+d>0$. The linearized system at $(0,\varphi^{-1}(m))$ is given by the matrix $\begin{pmatrix} m(q+1-p) & 0 \\ 0 & K(m)\end{pmatrix}$, with $K(m)=p(p-1)(\eta^2-m^2)/(1+(p-1)m^2)$. If $m=m_{1,d}$, then it is a source, and we find again the existence of an infinity of solutions. If $m=m_d$ or $m=m_{2,d}$, then $K(m)<0$, thus this point is a saddle point. Then in the phase plane (v,u), there exists precisely one trajectory defined near $-\infty$, such that $v>0$ and converging to $(0,m)$ at $-\infty$, and u/v converges to 0. $\square$

Remark. Suppose $f(w)=|w|^{q-1}w$, then we can study the critical case $(p-2)b>2(p-1)$ and $E(\eta)=d$: there exists an infinity of homoclinic trajectories $\mathcal{H}_1$ starting from $(0,0)$ in $\mathcal{Q}$ with an infinite slope and ending at $(0,0)$ with an infinite slope, and a unique homoclinic trajectory $\mathcal{H}_2$ starting from $(0,0)$ in $\mathcal{Q}$ with the slope η and ending at $(0,0)$ with the slope $-\eta$. Indeed using system (3.20) and setting $u=\varphi^{-1}(\eta)+z$, and $\zeta=(q+1-p)\eta z+v$, it can be written under the form

$$\zeta'=P(\zeta,v),\qquad\qquad v'=(q+1-p)\eta v+Q(\zeta,v),$$

where P and Q both start with quadratic terms. Moreover the quadratic part of $P(\zeta,v)$ is given by $p_{2,0}\zeta^2+p_{1,1}\zeta v+p_{0,2}v^2$, where by computation,

$$p_{2,0}=-\frac{p(p-1)}{q+1-p}\eta\varphi'^2(\varphi(\eta))(1+\eta^2)^{(p-2)/2}<0;$$

then the results follow from the description of sadle-node behaviour given in [8, Theorem 9.1.7].

Remark. In the case $b=1>-d$, we have a representation of the homoclinic trajectory : it corresponds to $K=0$ in (4.2). In the case $f(w)=|w|^{q-1}w$, in terms of u we obtain

$$u'=\frac{q+1-p}{p}(E(\varphi(u))-d),$$

which allows to compute u by a quadrature.

4.3 Period of the solutions

First we consider the sign changing solutions

Theorem 4.3 *Consider system (3.9) with $p>1$. For any $\nu>0$, consider the trajectory $\mathcal{T}_{[(0,\nu)]}$ which goes through $(0,\nu)$. Let $T(\nu)$ be its least period . Then T is decreasing on $(0,\infty)$.*

(i) If $b+d \le 0$, and E is increasing or $d < \min E$, it decreases from T_d to 0, where

$$T_d = 4\int_0^{\infty} \frac{du}{E(\varphi(u)) - d} = 4\int_0^{\pi/2} \frac{1+(p-1)\tan^2\theta}{(p-1)\tan^2\theta - b - d\cos^{p-2}\theta} d\theta, \tag{4.3}$$

and T_d is finite if and only if $b+d<0$. If $b<0=d$, then

$$T_0 = 2\pi \frac{(p-1)\gamma+1}{(p-1)\gamma(\gamma+1)}, \qquad \gamma = \sqrt{|b|/(p-1)}). \tag{4.4}$$

(ii) If $b+d>0$ or $b+d \le 0$ and $d \ge \min E$, it decreases from ∞ to 0.

Proof. **(i) Monotonicity of** T : consider the part of the trajectories $\mathcal{T}_{[(0,\nu)]}$ located in $\mathcal{Q}$, given by (w_ν, y_ν). We have shown that u is decreasing with respect to τ from ∞ to 0, then $E(\varphi(u)) + h(w_\nu(u)) - d > 0$ and w_ν can be expressed in terms of u, and

$$T(\nu) = 4\int_0^{\infty} \frac{du}{E(\varphi(u)) + h(w_\nu(u)) - d}.$$

Let $\lambda > 1$. Since the trajectories $\mathcal{T}_{[(0,\nu)]}$ and $\mathcal{T}_{[(0,\lambda\nu)].}$ have no intersection point, then $w_{\lambda\nu}(u) > w_\nu(u)$ for any $u>0$, and h is nondecreasing, thus $T(\lambda\nu) < T(\nu)$, thus T is decreasing.

(ii) Behaviour near ∞ : Let $\nu_n \ge 1$, such that $\lim \nu_n = \infty$. Observe that for fixed u, for any integer $n \ge 1$, there exists a unique $\tilde{\nu}_n > 0$ (depending on u), such that $w_{\tilde{\nu}_n}(u) = n$; let $\hat{\nu}_n = \max(\tilde{\nu}_n, n)$. Then $h(w_{\hat{\nu}_n}(u)) \ge h(n)$ thus $h(w_{\hat{\nu}_n}(u))$ converges to ∞; since $\nu \mapsto h(w_\nu(u))$ is nondecreasing then $h(w_{\nu_n}(u))$ converges to ∞, and $T(\nu_n)$ converges to 0, from the Beppo-Levi theorem.

(iii) Behaviour near 0 :

• First assume $b+d \le 0$, and E is increasing, or $d < E(\eta)$. Then all the orbits are of the type $\mathcal{T}_{[(0,\nu)]}$. Let $\sigma_n \in (0,1)$, such that $\lim \nu_n = 0$. For fixed u, for any integer $n \ge 1$, there exists a unique $\bar{\nu}_n > 0$ (depending on u), such that $w_{\bar{\nu}_n}(u) = 1/n$; let $\check{\nu}_n = \min(\bar{\nu}_n, 1/n)$. Then $h(w_{\check{\nu}_n}(u) \le h(1/n)$, thus $h(w_{\check{\nu}_n}(u))$ converges to 0, thus again $h(w_{\nu_n}(u))$ converges to 0. Then $T(\nu_n)$ converges to T_d given by (4.3), from the Beppo-Levi theorem. If $b+d<0$, then T_d is finite: indeed near ∞, $E(\varphi(u)) = (p-1)u^{p/(p-1)}(1+o(1))$; if E is increasing, then $E(\varphi(u)) - d > -(b+d) > 0$; if $d < E(\eta)$, then $E(\varphi(u)) - d \ge E(\eta) - d > 0$.

If $b+d=0$ and E is increasing, then $T_d = \infty$: indeed near 0, $E(\varphi(u)) - d = u^2(E''(0)/2 + o(1))$ and $E''(0) = 2(p-1) - (p-2)b$; if $(p-2)b = 2(p-1)$, then $E(\varphi(u)) - d = (p(p-1)/4)u^4(1+o(1))$; in any case the integral is divergent.

When $b<0=d$, one can compute T_0 :

$$\frac{T_0}{4} = \int_0^{\infty} \frac{du}{E(\varphi(u))} = \int_0^{\infty} \frac{\phi'(\xi)\, d\xi}{E(\xi)} = \int_0^{\infty} \frac{1+(p-1)\xi^2}{(|b|+(p-1)\xi^2)(1+\xi^2)} d\xi$$
$$= \frac{\pi}{2} + (\frac{1}{p-1} - \gamma^2)\int_0^{\infty} \frac{ds}{(\gamma^2+s^2)(1+s^2)} = \frac{\pi}{2}(1 + \frac{1-(p-1)\gamma^2}{(p-1)\gamma(\gamma+1)}).$$

Hence (4.4) holds.

• Next assume $d > E(\eta)$. Considering ν_n as above, for any fixed u such that $\varphi(u) > m_2$, there exists a unique $\bar{\nu}_n > 0$ (depending on u), such that $w_{\bar{\nu}_n}(u) = 1/n$. Then as above

$$\int_{\varphi^{-1}(m_2)}^{\infty} \frac{du}{E(\varphi(u)) + h(w) - d} \to \int_{\varphi^{-1}(m_2)}^{\infty} \frac{du}{E(\varphi(u)) - d},$$

and the last integral is infinite, because the slope of E at m_2 is finite; as a consequence, $T(\sigma)$ tends to ∞. If $d = E(\eta)$, the same proof still works with m_2 replaced by η : the integral is still divergent; the denominator is of order 2 in $u - \varphi^{-1}(\eta)$; near 0,

$$E(\varphi(u)) - d = \frac{1}{2}E''(\eta)(\varphi(u) - \eta)^2(1 + o(1)) = \frac{1}{2}E''(\eta)(\varphi(u) - \eta)^2(1 + o(1),$$

and

$$E''(\eta) = 2p(p-1)\eta^2(1+\eta^2)^{(p-2)/2} > 0.$$

At last suppose $b + d > 0$; the same proof with m_2 replaced by m shows that $T(\sigma)$ converges to ∞ as σ tends to 0, since the slope of E at $m = E^{-1}(d)$ is finite. □

The monotonicity of the period function proved above is a more general property, as shown in the next lemma.

Lemma 4.4 *Consider a system of the form*

$$w' = F(w, y), \qquad y' = G(w, y)$$

where $F, G \in C^1(\mathbb{R}^2 \backslash (0,0))$. Assume that F (resp. G) is odd with respect to y (resp. x) and even with respect to x (resp. y), with $F(w, y) > 0$ in $\mathcal{Q}$. Assume that for any $(w, y) \in \mathcal{Q}$, and any $\lambda > 0$,

$$\frac{\partial}{\partial\lambda}\left(\frac{F(\lambda w, \lambda y)}{\lambda}\right) \geq 0 \ (resp. \leq 0) \quad and \ \frac{\partial}{\partial\lambda}\left(\frac{G(\lambda w, \lambda y)}{\lambda}\right) < 0 \ (resp. > 0). \tag{4.5}$$

Assume also that for any σ in an interval (σ_1, σ_2) with $0 < \sigma_1 < \sigma_2$ the trajectory $\mathcal{T}_{[(0,\sigma)]}$ passing through $(0, \sigma)$ (necessarily entering $\mathcal{Q}$ since $F(0, \sigma) > 0$) leaves $\mathcal{Q}$ transversally in a finite time $T(\sigma)/4$ at some point $(c(\sigma), 0)$ (thus $G(c(\sigma), 0) < 0$). Then from the symmetries, $\mathcal{T}_{[(0,\sigma)]}$ is a closed orbit surrounding $(0, 0)$, with period $T(\sigma)$. Then T is decreasing (resp. increasing) with respect to σ on (σ_1, σ_2).

Remark. We can notice the condition on F is equivalent to $F(\lambda w, \lambda y) \geq \lambda F(w, y)$ for any $\lambda > 1$; the second condition implies that for any $\lambda > 1$,

$$G(\lambda w, \lambda y) < \lambda G(w, y) \ (\text{resp.} G(\lambda w, \lambda y) > \lambda G(w, y)),$$

but is actually stronger.

Proof of Lemma 4.4. In polar coordinates (ρ, θ) in $\mathcal{Q}$, we get

$$\rho' = F\cos\theta + G\sin\theta, \qquad \theta' = \frac{1}{\rho}(G\cos\theta - F\sin\theta).$$

At each point τ where $\theta'(\tau) = 0$, we find by computation

$$\rho\theta''(\tau) = \left(\frac{\partial G}{\partial \rho}\cos\theta - \frac{\partial F}{\partial \rho}\sin\theta\right)\rho'(\tau) = \frac{F}{\cos\theta}\left(\frac{\partial G}{\partial \rho}\cos\theta - \frac{\partial F}{\partial \rho}\sin\theta\right).$$

But (4.5) is equivalent to $\partial F/\partial\rho \geq F/\rho$ and $\partial G/\partial\rho < G/\rho$ (resp $>$), thus

$$\rho\theta''(\tau) < \frac{F}{\rho\cos\theta}(G\cos\theta - F\sin\theta) = 0 \quad (\text{resp.} >)$$

In any case θ'' has a constant sign. But $\theta'(0) = -F(0,\sigma) < 0$ and $\theta'(\sigma) = G(c(\sigma), 0) < 0$ thus we get a contradiction by considering the first (resp. the last) point where $\theta'(\tau) = 0$, which satisfies $\theta''(\tau) \geq 0$ (resp. ≤ 0). Thus θ is decreasing from $\pi/2$ to 0. Then the curves can be represented in function of θ by $(\rho(\sigma,\theta), \theta(\sigma))$, and

$$T(\sigma) = 4\int_0^{\pi/2} \frac{d\theta}{H(\rho(\sigma,\theta),\theta)}$$

with

$$H(\rho,\theta) = \frac{1}{\rho}(F(\rho\cos\theta, \rho\sin\theta)\sin\theta - G(\rho\cos\theta, \rho\sin\theta)\cos\theta)$$

Let $\lambda > 1$. Since the trajectories $\mathcal{T}_{[(0,\sigma)]}$ and $\mathcal{T}_{[(0,\lambda\sigma)]}$. have no intersection point, then $\rho(\lambda\sigma,\theta) > \rho(\sigma,\theta)$ for any $\theta \in (0,\pi/2)$; by hypothesis for fixed θ, the function $\rho \mapsto F(\rho\cos\theta, \rho\sin\theta)/\rho$ is nondecreasing (resp. nonincreasing) $\rho \mapsto G(\rho\cos\theta, \rho\sin\theta)/\rho$ is decreasing (resp. increasing), thus $H(\rho(\lambda\sigma,\theta),\theta) > H(\rho(\sigma,\theta),\theta)$, thus $T(\lambda\sigma) < T(\sigma)$ (resp. $>$) ; then T is decreasing (resp. increasing). □

Next we consider the positive solutions w of equation (3.4).

Proposition 4.5 *Let $p > 1, b + d > 0$. Consider the trajectories $\mathcal{T}_{[(\mu,0)]}$ in the phase plane (w, y) which goes through $(\mu, 0)$, $\mu \in (0, a)$. Let $T^+(\mu)$ be their least period. Then*

$$\lim_{\mu\to 0} T^+(\mu) = \infty, \qquad \lim_{\mu\to a} T^+(\mu) = \frac{2\pi}{\sqrt{ah'(a)}}.$$

In particular if $f(w) = |w|^{q-1}w$, then $\lim_{\mu\to a} T^+(\mu) = 2\pi/(q + 1 - p)(b + d)$.

Proof. We notice that the trajectory $\mathcal{T}_{[(\mu,0)]}$ intersects the line $y = 0$ at $(\mu, 0)$ and another point $(g(\mu), 0)$, with $\mu < a < g(\mu)$, and g is decreasing.

(i) Behaviour near a : when μ tends to a, then also $g(\mu)$ tends to a. Indeed for any small $\varepsilon > 0$, then $g(\mu) - a < \varepsilon$ as soon as $\mu - a < \min(\varepsilon, a - g^{-1}(a + \varepsilon))$. Since $\xi = \varphi(u)$ varies from 0 to 0 in $\mathcal{Q}$ it has a maximal point ξ^*, where $u' = 0$, thus $E(\xi^*) = h(w^*)$. When μ tends to a, then $h(w^*)$ tends to b, thus ξ^* tends to $E^{-1}(b) = 0$, thus also $\max_{y\in\mathcal{T}_{[(\mu,0)]}} |y|$ tends to 0 Using the matrix of the linearization at P_0, and passing in polar coordonates near $(a, 0)$, $w = a + r\cos\eta$, $y = \sqrt{ah'(a)}r\sin\eta$, then r tends to 0 as μ tends to a, and one finds $\eta' = -\sqrt{ah'(a)} + R/r$, where R involves the derivatives of G of order 2 , which are bounded near the point $(a, 0)$, thus R/r^2 is bounded. Then η' tends to $-\sqrt{ah'(a)}$, thus $T^+(\mu)$ tends to $2\pi/\sqrt{ah'(a)}$.

(ii) Behaviour near 0 : on $\mathcal{T}_{[(\mu,0)]}$, the function u is increasing up to a maximal value $u^*(\mu)$, and then decreasing, and u^* is a nonincreasing function of μ, because two different curves have no intersection. Let $\mu_n \in (0,a)$, such that $\lim \mu_n = 0$. For any n there exists $\tilde{\mu}_n \in (0,a)$ such that the orbit has a point above the line $y = \varphi^{-1}(m)(1-1/n)w$, let $\hat{\mu}_n = \min(\mu_n, 1/n)$. Then $u^*(\hat{\mu}_n) \geq \varphi^{-1}(m)(1-1/n)$, thus $u^*(\mu_n)$ tends to m ; then from the Beppo-Levy theorem

$$\liminf T^+(\mu) \geq \lim \int_{u^*(\mu)}^{\infty} \frac{du}{E(\varphi(u)) - d + h(w(\mu,u))} = \int_m^{\infty} \frac{du}{E(\varphi(u)) - d + h(w(u))}$$

where w is the solution defining $\mathcal{H}$, and this integral is infinite. □

Remark. Here the question of the monotonicity of the period is quite hard to answer, even for $p = 2$, where it is solved by using the first integral, see [2]. It is open in the general case. In dynamical systems with a center, the qualitative behaviour of the period function is not completely understood, even in quadratical ones: one can contruct such a system with a center whose associated period function is not monotonous, and even with at least two critical points, see [4] and [5].

Remark. In the case $b = 1$, we can compute theoretically the period T^+ by using the first integral (4.1). The stationary point $P_0 = (h^{-1}(1), 0)$ is obtained for $K_a = a^p/p - \mathcal{F}(a) > 0$ (in case of a power, $K_a = (q+1-p)/p(q+1)$). The positive solutions correspond to trajectories $\mathcal{T}_K$ with $K \in (0, K_a)$, intersecting the axis $y = 0$ at points $(w_1, 0)$, $(w_2, 0)$ with $w_1 < a < w_2$ defined by $w_i^p/p - \mathcal{F}(w_i) = K$, and the period is given by

$$T^+ = 2\int_{w_1}^{w_2} \frac{dw}{wE^{-1}(-p\frac{K+\mathcal{F}(w)}{w^p})}.$$

This formula does not allow us to prove the monotonicity of the period function for $p \neq 2$.

In the case $f(w) = |w|^{q-1} w$, we can solve the problem in a particular case: $b = 1$ and $q = 2p - 1$, by using the equation (3.21) relative to u :

Proposition 4.6 *Suppose that $f(w) = |w|^{q-1} w$, and $b = 1$ and $q = 2p - 1$, $p > 1$ and $d + 1 > 0$. If $p > 2$ or $d + 1 < 1/(2-p)$, then T^+ is decreasing on $(0,a)$.*

Proof. In equation (3.21) the coefficient B is zero, thus

$$u'' = (q+1-p)(E(\varphi(u) - d)\varphi(u) = (q+1-p)(-(1+d) + p\Psi.(u))\varphi(u)$$

Henceforth

$$\frac{1}{q+1-p} u''u' = -(1+d)\Psi'(u)u' + p\Psi(u)\Psi'(u)u',$$

from which expression we derive the first integral,

$$\frac{1}{q+1-p} u'^2 = C - \mathcal{U}(u)), \qquad \mathcal{U} = \mathcal{M} \circ \Psi, \qquad \mathcal{M}(t) = 2(1+d)t - pt^2. \tag{4.6}$$

From (4.6) the curves $\mathcal{S}$ are symmetric with respect to the axis $u' = 0$. The times for going from $u = 0$ to $u = u^*$ and from u^* to 0 are equal, and u^* is given by $C = \mathcal{M}(\Psi(u^*))$.The ccomputation of the period is reduced to the part relative to the first quadrant. Here we follow the method of [2]: we get

$$T^+(u^*) = 4\int_0^{u^*} \frac{d\eta}{\sqrt{\mathcal{U}(u^*) - \mathcal{U}(\eta)}} = 4\int_0^1 \frac{u^* ds}{\sqrt{\mathcal{U}(u^*) - \mathcal{U}(su^*)}}.$$

Then

$$\frac{dT^+(u^*)}{du^*} = 4\int_0^1 \frac{(\Theta(u^*) - \Theta(su^*))ds}{(\mathcal{U}(u^*) - \mathcal{U}(su^*))^{3/2}}, \qquad \text{with} \qquad \Theta(u^*) = \mathcal{U}(u^*) - u^*\mathcal{U}'(u^*)/2,$$

and

$$2\frac{d\Theta(u^*)}{du^*} = 2\Theta'(u^*) = \mathcal{U}'(u^*) - u^*\mathcal{U}''(u^*).$$

In the interval of study, $\varphi(u^*) < E^{-1}(d)$, $(E \circ \varphi)(u) < d$ from (3.12), thus $\Psi(u) < (1+d)/p$, and $\mathcal{M}$ is increasing for $0 < t < (1+d)/p$, thus $\mathcal{U}' > 0$. Then at any point u, $\Theta'(u) > 0 \Longleftrightarrow (\mathcal{U}'/u)' < 0$. Now

$$\frac{\mathcal{U}'(u)}{2pu} = \frac{(-E(\varphi(u)) + d)\varphi(u)}{u} = 1 - (p-1)\varphi^2(u) + d(1+\varphi^2(u))^{(2-p)/2},$$

hence $(\mathcal{U}'/u)' = 2X(u)\varphi(u)\varphi'(u)$, with

$$X(u) = -(p-1) + (2-p)d(1+\varphi^2(u))^{-p/2},$$

and $d > E(\varphi(u))$; it implies $X(u) < 0$ if $p > 2$ or $p < 2$ and $d < (p-1)/(2-p)$. Henceforth Θ is increasing, and the same holds for P as a function of u^*. Finally u^* is decreasing with respect to μ, and consequently P is decreasing with respect to μ. $\square$

Remark. When $p = 2$, and $q = 2p-1 = 3$, equation (3.21) reduces to $u'' = -2u + 2u^3$, which, surprisingly, is an equation correponding to the other sign, and (3.4) reduces to $w'' - w + w^3 = 0$. Here all the solutions can be expressed in terms of elliptic integrals, see [2].

4.4 Returning to the initial problem

Proof of Theorem 2. Here $\beta = \beta_q = p/(q+1-p)$, $\lambda = \lambda_q$ is given by (1.12) and $c_q = \beta_q^{p-2}\lambda_q$ by (1.13). And $\omega(\sigma) = \beta_q^{\beta_q} w(\beta_q\sigma)$ from (3.3), and $b = -\lambda_q/\beta_q^2 = -c_q/\beta_q^p$ and $d = c/\beta_q^p$ from (3.5), $f(s) = g(s) = |s|^{q-1} s$, thus $h(s) = |s|^{q-p} s$. Thus $c > c_q$ is equivalent to $b + d > 0$, and then the constant solutions $w \equiv \pm(b+d)^{1/(q-p+1)}$ of (3.4) correspond to the constant solutions $\omega \equiv \pm (c - c_q)^{1/(q+1-p)}$ of equation (1.11). For any integer $k \geq 1$, we search periodic solutions ω of smallest period $2\pi/k$, or equivalently solutions w of period $T_k = 2\pi\beta_q/k$. From (3.15), the function E is increasing. First consider the sign changing solutions: if $c \geq c_q$, then from Theorem 4.3, the period function T of w is decreasing from ∞ to 0, hence for any $k \geq 1$ it takes precisely once the value T_k. If $c < c_q$, then T decreases from T_d given by (4.3) to 0, thus it takes once the value T_k for any $k > M_q = T_d/2\pi\beta_q$ given at (1.15). Next consider the positive solutions: from Proposition 4.5, the period function of w takes any value between ∞ and $2\pi/\sqrt{(q+1-p)(b+d)}$, thus it takes the value T_k for any $k < (p\beta_q^{1-p}(c-c_q))^{1/2}$, which achieves the proof. $\square$

In the case of equation (1.11) without potential (that means $c = 0$), we deduce the following:

Corollary 4.7 *Assume $p > 1$, $q > p-1$, and $c = 0$.*

(i) Then $\mathcal{E}$ is given by (1.14), where $k_q = 1$ if $p < 2$ and $q \geq 2(p-1)/(2-p)$, and $k_q > M_q$ if $p \geq 2$ or ($p < 2$ and $q < 2(p-1)/(2-p)$), where $M_q = 2/(q-1)$ for $p = 2$, and

$$M_q = \frac{(p-2)m_q}{((p-1)m_q+1)(m_q-1)}, \ m_q = \sqrt{\frac{(2(p-1)+(p-2)q}{p(p-1)}} = \sqrt{1+\frac{p-2}{(p-1)\beta_q}} \ \text{ for } p \neq 2. \tag{4.7}$$

(ii) If $p \geq 2$ or ($p < 2$ and $q < 2(p-1)/(2-p)$), then $\mathcal{E}^+ = \varnothing$. If $p < 2$ and $q \geq 2(p-1)/(2-p)$, then $\mathcal{E}^+ = \left\{(-c_q)^{1/(q+1-p)}\right\}$.

Proof. Here $c_q < 0$ is equivalent to $p < 2$ and $q > 2(p-1)/(2-p)$. And $M_q = T_0/2\pi\beta_q$ can be computed from (4.4), which gives (4.7). Moreover in any case $c_q + \beta_q^{p-1}/p = \beta_q^p(p-1)(q+1)/p^2 > 0$ thus there exist no positive nonconstant periodic solutions. $\square$

Proof of Corollary 1. Let S be a sector on S^1 with opening angle $\theta \in (0, 2\pi)$. From [9, Th 3.3], β_S is the positive solution of equation

$$\phi(\beta_S) = \left(1+\frac{1}{k}\right)^2\left(\beta_S^2 + \frac{p-2}{p-1}\beta_S - (\beta_S+1)^2\right) = 0,$$

where $k = \pi/\theta \geq 1$. From Corollary 4.7 (applied without assuming that k is an integer) we distinguish two cases:

(i) $p < 2$ and $q \geq 2(p-1)/(2-p)$. Then there always exists a solution of the Dirichlet problem in S. Notice that $0 < \beta_q \leq (2-p)/(p-1)$, thus $\phi(\beta_S) < 0$ and consequently $\beta_q < \beta_S$.

(ii) $p > 2$ or $p < 2$ and $q < 2(p-1)/(2-p)$. The existence is equivalent to $k > M_q$ (see (4.7)). It means

$$\left(1+\frac{1}{k}\right)^2 < \left(\frac{(p-1)m_q^2-1}{m_q(p-2)}\right)^2 = \frac{(\beta_q+1)^2}{\beta_q^2 m_q^2} = \frac{(\beta_q+1)^2}{\beta_q(\beta_q+(p-2)\beta_q/(p-1))}.$$

Thus $\phi(\beta_q) < 0$. Equivalently, $\beta_q < \beta_S$. $\square$

5 The case $p = 1$

5.1 Existence of a first integral

As shown in Lemma 3.1, we can reduce the study to

$$\frac{d}{d\tau}\left(\frac{w'}{\sqrt{w^2+w'^2}}\right) - b\frac{w}{\sqrt{w^2+w'^2}} + f_1(w) - d\,|w|^{-1}\,w = 0, \tag{5.8}$$

where f_1 satisfies (3.8); in particular we are interessed by the case $f_1(s) = s$.

Here the problem is variational: if $S_1(w)$ is any primitive of $w \mapsto |w|^{b-1} f_1(w)$ and $R(w) = |w|^b/b$ if $b \neq 0$, $R(w) = \ln|w|$ if $b = 0$, then (5.8) is the Euler equation of the functional

$$\mathcal{H}(w, w') = |w|^{b-1}\sqrt{w^2+w'^2} - S_1(w) + dR(w).$$

Thus we have the Painlevé integral

$$\frac{|w|^{b+1}}{\sqrt{w^2+w'^2}} - S_1(w) + dR(w) = C. \tag{5.9}$$

The system (3.9)

$$\begin{cases} w' = y \\ y' = G(w,y) = \dfrac{bw^3 + (b+1)w\,y^2 - (f(w) - d|w|^{-1}w)(w^2+y^2)^{3/2}}{w^2}, \end{cases}$$

is singular on the line $w = 0$. System (3.13) reduces (for $w > 0$) to

$$\begin{cases} w' = w\varphi(u) = w\frac{u}{\sqrt{1-u^2}} \\ u' = b\sqrt{1-u^2} - f_1(w) + d. \end{cases} \tag{5.10}$$

In the case $f(w) = w$, the equation in u'' is

$$u'' = (1-b)\frac{u}{\sqrt{1-u^2}}u' - bu - d\frac{u}{\sqrt{1-u^2}}. \tag{5.11}$$

5.2 Existence of periodic solutions

From the Painlevé integral (5.9), we can describe the solutions, by using the phase plane (w, y). Since a complete description is rather long, we reduce it to the research of periodic solutions.

Proposition 5.1 *Let $p = 1$, and consider equation (5.8).*

(i) If $d \neq 0$, there is no periodic sign changing solution. If $d = 0$ there exists such a solution if and only if $b > -1$, and then it is unique (up to a translation).

(ii) There exists periodic positive solutions if and only if $b + d > 0$.

(iii) Suppose moreover that $f_1(w) = w$. Then the sign changing solution is given by

$$w(\tau) = (b+1)\cos(\tau - \tau_1),$$

of period 2π. The closed orbits $\mathcal{T}_{[(\mu,0)]}$ of the periodic solutions intersect the axis $y = 0$ at a first point $(\mu, 0)$ such that $\mu < a = b + d$, and μ describes $\mu \in (\bar{\mu}, a)$ with $\bar{\mu} = 0$ if $d \leq 0$, and $\bar{\mu} > 0$ if $d > 0$; it is given by (5.16),(5.14),(5.15).

Proof. By symmetry we reduce the study to the case $w \geq 0$ and the integral (5.9) takes the form

$$w^b\sqrt{1-u^2} - S_1(w) + dR(w) = C \tag{5.12}$$

where we take $S_1(w) = \int_0^w s^{b-1} f_1(s)ds$ if $b > -1$ and $S_1(w) = \int_1^w s^{b-1} f_1(s)ds + 1/(b+1)$ if $b < -1$, and $S_1(w) = \int_1^w s^{-2} f_1(s)ds$ if $b = -1$.

(i) The curves in the phase plane (w, y) are given for $w > 0$ by

$$\begin{aligned} y^2 &= (\tfrac{w^{b+1}}{C - dR(w) + S_1(w)})^2 - w^2 \\ &= \frac{w^2\left(w^b + dR(w) - S_1(w) - C\right)(w^b - dR(w) + S_1(w) + C)}{\left(S_1(w) + C - dR(w)\right)^2}, \end{aligned}$$

which defines $\pm y$ in function of w. If there exists a sign changing periodic solution, the trajectory intersects the axis $w = 0$ at some point $(0, \ell)$ with $\ell \geq 0$, thus y needs to ℓ as w tends to 0. From (5.12), it is impossible if $b \leq -1$. Assume $d \neq 0$; if $-1 < b$, then near $w = 0$, in any case $y^2 \leq (b^2/d^2 + 1)w^2$, thus $\ell = 0$ and w'/w is bounded, thus the maximal interval of existence is infinite, and we reach a contradiction. If $d = 0$, and $C \neq 0$, then $y^2 = -w^2(1 + o(1))$, which is impossible. If $d = C = 0$, then $y^2 = w^2(w^{2b}/S_1^2(w) - 1)$; observing that the function $w \mapsto \chi(w) = w^{-b}S_1(w)$ is increasing from 0 to ∞, the curve intersects the two axis at $(0, b+1)$ and $(\chi^{-1}(1), 0)$ and this corresponds to a closed orbit.

(ii) If we look at the intersection points of any trajectory in the phase plane with the axis $y_1 = 0$, we find that they are given by $H(w) = C$, where

$$H(w) = w^b + dR(w) - S_1(w).$$

Then $H'(w) = w^{b-1}(b + d - f_1(w))$. If $b + d \leq 0$, then H is decreasing, thus there exist no positive periodic solutions. If $b + d > 0$, the function H is increasing on $(0, a)$ up to a maximum M, and decreasing on (a, ∞). The stationary point $(a, 0)$ with $a = f_1^{-1}(b+d)$ corresponds to $C = M$. If $b > 0$, then $\lim_{w\to 0} H = 0$, if $b \leq 0$, then $\lim_{w\to 0} H = -\infty$. Equation $H(w) = C$ has two roots $0 < w_1 < w_2$, if and only if $C \in (\max\{\lim_{w\to 0} H, \lim_{w\to\infty} H\}, M)$. Moreover if there is a trajectory going through $(w_1, 0)$ and $(w_2, 0)$ defining

$$K(w) = dR(w) - S_1(w) = H(w) - w^b,$$

one has $K(w) < C$ on (w_1, w_2), thus $C > M' = \max K = K(f_1^{-1}(d))$. Reciprocally, if

$$\max\left\{\lim_{w\to 0} H, \lim_{w\to\infty} H, M'\right\} < C < M, \quad M = H(f^{-1}(b+d)), \quad M' = K(f^{-1}(d)) \tag{5.13}$$

then there exists a closed orbit going through $(w_1, 0)$ and $(w_2, 0)$.

(iii) The sign changing solution is given by $w^2 + w'^2 = (b+1)^2$ and its trajectory is a circle of center 0 and radius $b+1$; for $w > 0$, $w = (b+1)\sqrt{1-u^2} = b\sqrt{1-u^2} - u'$, thus $u' = -\sqrt{1-u^2}$, and $\theta' = -1$, then $w(\tau) = (b+1)\cos(\theta - \tau_1)$, periodic solution with period 2π. Now consider the positive periodic solutions. Here $a = b + d$, and

$$H(w) = \begin{cases} (1 + d/b)w^b - w^{b+1}/(b+1), & \text{if } b \neq 0, -1, \\ 1 + d \ln w - w, & \text{if } b = 0, \\ (1-d)w^{-1} - \ln w, & \text{if } b = -1, \end{cases} \tag{5.14}$$

$$\begin{cases} M = a^{b+1}/b(b+1), & M' = (d^+)^{b+1}/b(b+1)), & \text{if } b \neq 0, -1, \\ M = 1 + d \ln d - d, & M' = d \ln d - d, & \text{if } b = 0. \\ M = -1 - ln(d-1), & M' = -1 - \ln d)), & \text{if } b = -1, \end{cases} \tag{5.15}$$

If $d \leq 0$, thus $b > 0$, then any $C \in (0, M)$ corresponds to a closed orbit, thus for any $\mu \in (0, a_1)$, one has a closed orbit passing through $(\mu, 0)$, of period still denoted by $T^+(\mu)$. If $d > 0$, in any case, any $C \in (M', M)$ corresponds to a closed orbit. If $-1 < b < 0$, then H is increasing on $(0, a)$ from $-\infty$ to $M < 0$, and then decreasing on (a, ∞) from M to $-\infty$. If $b < -1$, then $M > M' > 0$. If $b < -1$, then $d > 1$, and $\lim_{w \to 0} H = -\infty$, $\lim_{w \to \infty} H = 0$, $0 < M' < M$. If $b = 0$, thus $d > 0$, then $\lim_{w \to 0} H = -\infty$, then any $C \in (M-1, M)$ corresponds to a closed orbit. Then H is increasing on $(0, d)$ from $-\infty$ to $M = 1 + d \ln d - d \geq 0$, $(M = 0 \Leftrightarrow d = 1$ and then decreasing on (a_1, ∞) from M to $-\infty$, and ..)); let $\bar{\mu} \in (0, b+d)$ be defined by

$$H(\bar{\mu}) = M', \tag{5.16}$$

thus for any $\mu \in (\bar{\mu}, a)$, one has a closed orbit passing through $(\mu, 0)$, of period still denoted by $T^+(\mu)$. If $-1 \leq b < 0$, thus $d > -b > 0$, then $\lim_{w \to 0} H = -\infty = \lim_{w \to \infty} H, M < 0$, and any $C \in (M', M)$ corresponds to a closed orbit (if $b = -1$, then $H(w) = (1-d)w^{-1} - \ln w$, $M = -1 - ln(d-1)$, $M' = -1 - \ln d)$. Returning to equation (1.11), the conclusion follows with $\bar{\mu}_q = \bar{\mu}^q$. □

5.3 Period of the solutions

Let $p = 1, b + d > 0$. Consider the equation (5.8). Let $T^+(\mu)$ be the least period of the periodic positive solutions on the orbit $\mathcal{T}_{[(\mu,0)]}$. As in the case $p > 1$, we have a general result:

$$\lim_{\mu \to a} T^+(\mu) = \frac{2\pi}{\sqrt{a f_1'(a)}}.$$

Next we study the variations of the period in the case of a power $f_1(w) = w$.

Theorem 5.2 *Let $p = 1, b + d > 0$ and $f_1(w) = w$. Then $\lim_{\mu \to a} T^+(\mu) = 2\pi/\sqrt{b+d}$. If $d < 0$, then $\lim_{\mu \to 0} T^+(\mu) = \infty$. If $d \geq 0$, then $\lim_{\mu \to \bar{\mu}} T^+(\mu) = \bar{T}^+$ is finite, and given by (5.18) if $b \neq 0, -1$, (5.19) if $b = 0$, or (5.20) if $b = -1$. If $d = 0$, then $\bar{T}^+ = \pi(1 + 1/b)$.*

Proof. **(i)** Let $b \neq 0, -1$. From (5.12), the solutions of (5.8) satisfy

$$w^b \sqrt{1-u^2} - \frac{w^{b+1}}{b+1} + \frac{d}{b} w^b = C,$$

thus

$$u' = b\sqrt{1-u^2} - w + d = -\sqrt{1-u^2} - \frac{d}{b} + \frac{C(1+b)}{w^b}.$$

Eliminating w in the two relations, we find that $Cb(b+1) > 0$ and $A = (Cb(1+b))^{1/(b+1)}$

$$\left(d + b\sqrt{1-u^2} - u'\right)^{b/(b+1)} \left(d + b\sqrt{1-u^2} + bu'\right)^{1/(b+1)} = A$$

When the half-part of $\mathcal{T}$ located in $\mathcal{Q}$ is described, thus u increases from 0 to some $u^* \in (0,1)$ corresponding to $u' = 0$, then $d + b\sqrt{1-u^{*2}} > 0$; now $d + b\sqrt{1-u^2}$ is monotone and positive at 0 and u^*, thus $d + b\sqrt{1-u^2} > 0$ everywhere. And $A = d + b\sqrt{1-u^{*2}} = w^*$ (value of w at u^*) Let

$$z = \frac{u'}{d + b\sqrt{1-u^2}} \quad \text{and} \quad G(s) = (1-s)^{b/(b+1)} (1+bs)^{1/(b+1)},$$

If $b > 0$, then $z \in (-1/b, 1)$; if $b < -1$ then $z \in (-\infty, 1)$; if $-1 < b < 0$ then $z \in (-\infty, 1/|b|)$, and

$$G(z) = \frac{A}{d + b\sqrt{1-u^2}}.$$

Since

$$G'(s) = -bs\,(1-s)^{-1/(b+1)} (1+bs)^{-b/(b+1)},$$

and

$$G''(s) = -b\,(1-s)^{-(b+2)/(b+1)} (1+bs)^{-(2b+1)/(b+1)},$$

it follows $G(0) = 1$ and 0 is a maximum if $b > 0$ and a minimum if $b < 0$; if $b > 0$, G increases on $(-1/b, 0)$ from 0 to 1 and decreases on $(0,1)$ from 1 to 0. If $b < 0$, G decreases on $(-\infty, 0)$ from ∞ to 1 and increases on $(0, \min(1, 1/|b|))$ from 1 to ∞. Thus it has two inverse functions $-L_1$ and L_2 : for $b > 0$, L_1 maps $(0,1)$ into $(0, 1/b)$ and L_2 maps $(0,1)$ into $(0,1)$; for $b < 0$, L_1 maps $(1,\infty)$ into $(0,\infty)$ and L_2 maps $(1,\infty)$ into $(0, \min(1, 1/|b|))$. Then

$$T^+ = T_1^+ + T_2^+, \qquad T_i^+ = \int_0^1 \psi_{i,u^*}(\lambda) d\lambda, \tag{5.17}$$

where

$$\psi_{i,u^*}(\lambda) = \frac{2u^*}{(d + b\sqrt{1-\lambda^2 u^{*2}}) L_i((d + b\sqrt{1-u^{*2}})/(d + b\sqrt{1-\lambda^2 u^{*2}}))}.$$

• First suppose $d < 0$ (thus $b > 0$); then one looks at the case where $C \to 0$, thus $\sqrt{1-u^{*2}} \to -d/b$, thus $u^* \to \bar{u} = \sqrt{1-d^2/b^2}$. Near $\bar{u}$,

$$\psi_{i,u^*}(\lambda) \geq \frac{2u^*}{b(\sqrt{1-\lambda^2 u^{*2}} - \sqrt{1-u^{*2}}) L_i(0)} \geq \frac{-d}{bL_i(0)(1-\lambda^2)},$$

therefore T_i^+ tends to ∞.

• Suppose $d \geq 0, b > 0$. 0ne looks at the case where $C \to M'$, thus $u^* \to 1$. There exists a constant $m > 0$ such that $0 \leq 1 - G(s) = G(0) - G(s) \leq m^2 s^2$ on $[-1/b, 1]$. Indeed $G'(0) = 0$ and G'' is bounded on $[-1/2b, 1/2]$, and on $[-1/b, -1/2b] \cup [1/2, 1]$ the quotient $(G(0) - G(s))/s^2$ is bounded. Thus $1/L_i(\eta) \leq m/\sqrt{1-\eta}$ on $[0, 1)$, hence taking $\eta = (d + b\sqrt{1 - u^{*2}})/(d + b\sqrt{1 - \lambda^2 u^{*2}})$, and computing $^+$

$$1 - \eta = \frac{b(1 - \lambda^2)u^{*2}}{(d + b\sqrt{1 - \lambda^2 u^{*2}})\left(\sqrt{1 - u^{*2}} + \sqrt{1 - \lambda^2 u^{*2}}\right)},$$

one finds $\psi_{i,u^*}(\lambda) \leq 4m/\sqrt{b\,(1 - \lambda^2)}$. From the Lebesgue theorem, as $u^* \to 1$, T^+ tends to the finite limit

$$\bar{T}^+ = \bar{T}_1^+ + \bar{T}_2^+, \qquad \bar{T}_i^+ = 2\int_0^1 \frac{d\lambda}{(d + b\sqrt{1 - \lambda^2})L_i(d/(d + b\sqrt{1 - \lambda^2}))} \tag{5.18}$$

in particular if $d = 0$, then $L_1(0) = 1/b, L_1(0) = 1$, thus $\bar{T}_{1,1}^+ = \pi$ and $\bar{T}_{1,2}^+ = \pi/b$.

• Suppose $b < 0$, thus $d > -b > 0$. Then again $C \to M'$, consequently $u^* \to 1$. The function

$$u^* \to Q(u^*, \lambda) = \eta = \frac{d + b\sqrt{1 - u^{*2}}}{d + b\sqrt{1 - \lambda^2 u^{*2}}} = 1 - \frac{b(1 - \lambda^2)u^{*2}}{\left(d + b\sqrt{1 - \lambda^2 u^{*2}}\right)\left(\sqrt{1 - u^{*2}} + \sqrt{1 - \lambda^2 u^{*2}}\right)}$$

is increasing on $(0, 1)$ from 1 to $d/(d + b\sqrt{1 - \lambda^2})$ and $d/(d + b\sqrt{1 - \lambda^2}) \leq d/(d + b) = \alpha$. There exists $m > 0$ such that $0 \leq G(s) - 1 \leq m^2 s^2$ on $[-L_1(\alpha), L_2(\alpha)]$, thus $1/L_i(\eta) \leq m/\sqrt{\eta - 1}$ on $(1/d/(d + b)]$. Thus as above, $\psi_{i,u^*}(\lambda) \leq 4m/\sqrt{|b|\,(1 - \lambda^2)}$, and T^+ tends to $\bar{T}^+$ defined at (5.18).

(iii) Assume $b = 0$. There exist periodic solutions for any $C \in (M - 1, M)$. The solutions are given by

$$\sqrt{1 - u^2} + H(w) = \sqrt{1 - u^2} + d \ln w - w = C$$

and $u' = -w + d$, thus u is maximal ($= u^*$) for $w = d$: therefore $\sqrt{1 - u^{*2}} + H(d) = C$, then

$$H(d - u') = H(d) + \sqrt{1 - u^{*2}} - \sqrt{1 - u^2}$$

and H has two inverse functions H_i from $(-\infty, H(d))$ into $(0, d)$ and (d, ∞), thus (5.17) holds with

$$\psi_{i,u^*}(\lambda) = \frac{2u^* d\lambda}{(d - H_i(H(d) + \sqrt{1 - u^{*2}} - \sqrt{1 - \lambda^2 u^{*2}})}$$

and $\xi = H(d) + \sqrt{1 - u^{*2}} - \sqrt{1 - \lambda^2 u^{*2}} = H(d) - k = H(d + h)$ stays in $(M - 1, M) = (H(d) - 1, H(d))$, and $H(d+h) - H(d) \geq -m^2 h^2$ for $H(d+h) \in (M - 1, M)$, thus $H(d) - \xi = k \leq m^2(d - H_i(\xi))^2$, thus

$$\psi_{i,u^*}(\lambda) \leq \frac{2m}{\sqrt{k}} = \frac{2m\left(\sqrt{1 - u^{*2}} + \sqrt{1 - \lambda^2 u^{*2}}\right)}{\sqrt{1 - \lambda^2}} \leq \frac{4m}{\sqrt{1 - \lambda^2}}.$$

Therefore, as $u^* \to 1$, T^+ tends to the finite limit

$$\bar{T}^+ = \bar{T}_1^+ + \bar{T}_2^+, \qquad \bar{T}_i^+ = 2\int_0^1 \frac{d\lambda}{(d - H_i(H(d) - \sqrt{1-\lambda^2})}. \tag{5.19}$$

(iv) Assume $b = -1$; then $d > 1$; let $B = -(C+1) \in (\ln(d-1), \ln d)$ then $B \to \ln d$ and

$$u' + w = d - \sqrt{1-u^2} = (B+1)w - w\ln w = H_B(w)$$

where H_B is increasing on $\left(0, e^B\right)$ from 0 to e^B and decreasing on $\left(e^B, \infty\right)$ from e^B to $-\infty$; it has two inverse functions $L_{B,i}$ from $\left(-\infty, e^B\right)$ into $(0, e^B)$ and (e^{B-1}, ∞); and $w^* = d - \sqrt{1-u^{*2}} = e^B$; then (5.17) holds with

$$\psi_{i,u^*}(\lambda) = \frac{2u^*}{d - \sqrt{1-\lambda^2 u^{*2}} - L_{B,i}(d - \sqrt{1-\lambda^2 u^{*2}}} = \frac{2u^*}{\left|H_{B-1}(L_{B,i}(d - \sqrt{1-\lambda^2 u^{*2}}))\right|}.$$

Because $H_{B-1}(e^B) = 0$, $H_{B-1}(x) - H_{B-1}(e^B) = H'_{B-1}(\xi)(x - e^B)$ and x ranges onto $(H_{B,1}(d-1), H_{B,2}(d-1)) := (x_{1,B}, x_{2,B})$, when $B \to \ln d$, $(x_{1,B}, x_{2,B}) \to (x_{1,\ln d}, x_{2,\ln d})$, it follows $\left|H'_{B-1}(\xi)\right| \geq 1/\mu > 0$ independent on B. Moreover $H_B(x) - H_B(e^B) = (1/2)H''_B(\xi)(x - e^B)^2 = -(1/2\xi)(x - e^B)^2$. Thus there exists $m > 0$ such that

$$H_B(x) - H_B(e^B) \leq m^2(x - e^B)^2 \leq m^2\mu^2 H^2_{B-1}(x).$$

Therefore, near $\ln d$, taking $x = L_{B,i}(d - \sqrt{1-\lambda^2 u^{*2}})$, one derive

$$\psi_{i,u^*}(\lambda) \leq \frac{2}{m\mu\sqrt{d - \sqrt{1-\lambda^2 u^{*2}} - e^B}} = \frac{2}{m\mu\sqrt{\sqrt{1-u^{*2}} - \sqrt{1-\lambda^2 u^{*2}}}} \leq \frac{4}{m\mu\sqrt{1-\lambda^2}}.$$

Consequently, as $u^* \to 1$, $T^+_{1,i}$ tends to the finite limit

$$\bar{T}^+ = \bar{T}_1^+ + \bar{T}_2^+, \qquad \bar{T}_{1,i}^+ = 2\int_0^1 \frac{d\lambda}{\left|H_{\ln d-1}(L_{\ln d,i}(d - \sqrt{1-\lambda^2}))\right|}. \tag{5.20}$$

□

Remark. In the case $d = 0, b \neq 1$, notice that T_1^+ and T_2^+ converges to $\pi/\sqrt{b}$ as μ tends to b (one can verify it by linearizing the equation in u) and respectively to π and π/b as μ tends to Thus if those functions are monotonous, they vary in opposite senses and it is not easy to get the sense of variations of their sum T_1^+. Moreover in the phase plane (w, y_1), as μ tends to 0, on can observe that the trajectory tends to a limit curve constituted of a segment $[(0,0),(0,b)]$ and half of the unique closed orbit surrounding $(0,0)$, circle of center 0 and radius $b+1$, which is covered in a time π

The case $b = 1$ is the most interesting for (5.8), since it corresponds to the initial problem (1.11). In that case we improve the results by showing the monotonicity of the period function:

Theorem 5.3 *Assume $b = 1$, $d > -1$. When $d = 0$ the function $T^+(\mu)$ is constant, equal to 2π, thus there exists an infinity of positive solutions w of (5.8), which all have a period 2π; they are given by*

$$w = \sqrt{1 - K^{*2} \sin^2 \tau} - K^* \cos \tau, \qquad \tau \in [-\pi, \pi], \quad K^* \in (0, 1). \tag{5.21}$$

When $d \neq 0$, then $T^+(\mu)$ is strictly monotone; if $d < 0$ it decreases from ∞ to $2\pi/\sqrt{1 + d}$; if $d > 0$ it increases from

$$\bar{T}^+ = 4 \int_0^1 \frac{du}{\sqrt{(d + \sqrt{1 - u^2})^2 - d^2}} = 4 \int_0^{\pi/2} \sqrt{\frac{\cos \theta}{\cos \theta + 2d}} d\theta \tag{5.22}$$

to $2\pi/\sqrt{1 + d}$.

Proof. • If $d = 0$, then $u'' = -u$, from (5.11), and $u = \sin \theta \in [0, 1)$, thus the positive solutions w are given in $\mathcal{Q}$ by

$$u = K^* \sin \tau, \qquad K^* \in [0, 1), \qquad \tau \in [0, \pi],$$

and the period T^+ is constant, equal to 2π. We obtain an infinity of positive solutions w, given explicitely by

$$w = \sqrt{1 - u^2} - u' = \sqrt{1 - K^{*2} \sin^2 \tau} - K^* \cos \tau, K^* \in (0, 1)$$

which intersect the axe $y = 0$ at points $w_i = (1 \mp K^*)$.

• In the general case $d > -1$, we find

$$\left(d + \sqrt{1 - u^2} - u'\right) \left(\sqrt{1 - u^2} + u' + d\right) = A^2$$

that means G is symmetric: $G(s) = \sqrt{1 - s^2}$, thus

$$u'^2 = (d + \sqrt{1 - u^2})^2 - A^2$$

$\sqrt{1 - u^{*2}} = A - d = \sqrt{2C} - d$; thus here $T_1^+ = T_2^+$, and

$$T^+ = 4 \int_0^1 \frac{d\lambda}{\sqrt{\Psi(u^{*2}, \lambda)}}, \quad \text{where} \quad \Psi(s, \lambda) = \frac{(d + \sqrt{1 - \lambda^2 s})^2 - (d + \sqrt{1 - s})^2}{s}.$$

We show that the period function is strictly monotone with respect to u^*. Because

$$s^2 \frac{\partial \Psi}{\partial s}(s, \lambda) = d(d + 1) \left(1/\sqrt{1 - s} - 1/\sqrt{1 - \lambda^2 s}\right) > 0,$$

we see that T^+ is increasing if $d < 0$ and decreasing if $d > 0$ (and we find again that it is constant if $d = 0$). Also μ can be expressed explicitely in terms of u^* by

$$\mu = d + 1 - \sqrt{(d + 1)^2 - (d + \sqrt{1 - u^{*2}})}.$$

Therefore μ is decreasing with respect to u^*, hence T^+ is decreasing with respect to μ if $d < 0$ and increasing if $d > 0$. □

5.4 Returning to the initial problem

Proof of Theorem 3. Here $\alpha_q = \beta_q = 1/q$, the substitution (3.6) takes the form $\omega(\sigma) = |w(\sigma)|^{1/q-1} w(\sigma)$, and thus $b = 1$, and $d = c$ from (3.7). Then the existence of sign changing solutions of (1.11) is given by Proposition 5.1. The constant solutions exist whenever $c+1 > 0$. Searching positive solutions of smallest period $2\pi/k$. From Theorem 5.2 and Theorem 5.3, If $c < 0$ the period function T^+ decreases from ∞ to $2\pi/\sqrt{1+c} > 2\pi$, thus there exist no solution. If $c > 0$, T^+ increases from $\bar{T}^+$ given by (5.22) to $2\pi/\sqrt{1+c}$, thus it takes once any intermediate value, which gives one solution (up to a translation) for any $k \in (k_1, k_2)$. If $c = 0$, the solutions ω_K are given explicitely by (5.21), and ω_0^+ is obtained from ω_0, showing that system (3.9) does not satisfy the uniqueness property at $(0,0)$. □

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